DR. GYURCSEK ISTVÁN

# Sinusoids and Phasors

***Sources and additional materials (recommended)***


- *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- *Simonyi K.: Villamosságtan. AK Budapest 1983, ISBN:9630534134*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*
- *Zombory L.: Elektromágneses terek. MK Budapest 2006, (www.electro.uni-miskolc.hu)*

- ❑ **Sinusoids and Phasors**
- ❑ Circuit Elements in the Phasor Domain
- ❑ Kirchhoff’s Laws in the Phasor Domain
- ❑ Application Example – Phase Shifters

# Sinusoids 1

(Homework: overview the math. functions)

Periodic function $v(t) = v(t + nT)$

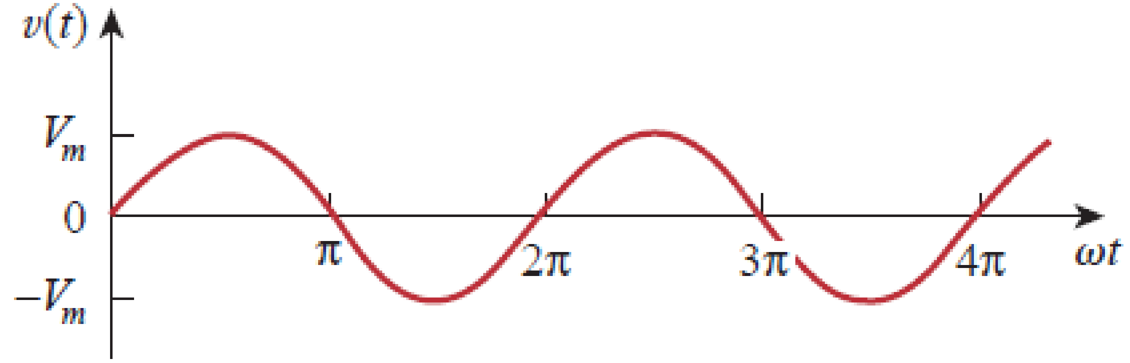
- → all *t*, → all integers *n*

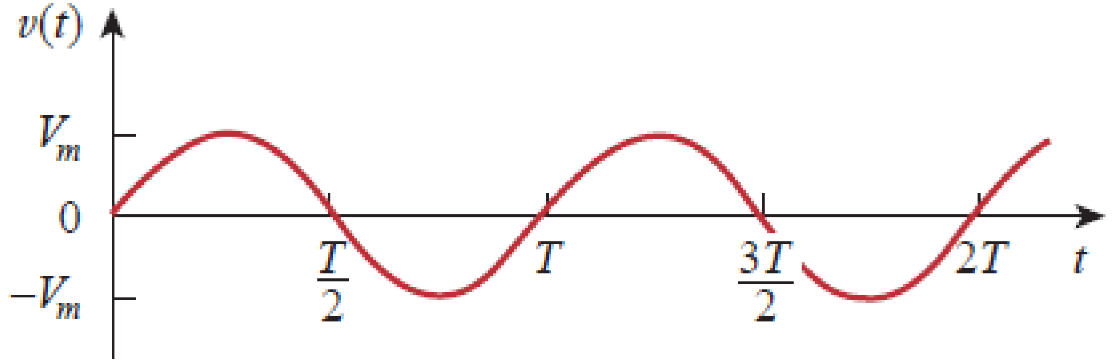
$$v(t) = V_m \sin \omega t = V_m \sin\{\omega(t + nT)\}$$

- Amplitude $\rightarrow V_m \ (V)$
- Argument $\rightarrow \omega t \ (rad)$
- Angular frequency $\rightarrow \omega \ ({}^{rad}/_{s})$

$$\omega T = 2\pi \qquad f = \frac{1}{T} \rightarrow \omega = 2\pi f$$

$$\left( wave\ propagation \rightarrow \lambda = c \cdot T = \frac{c}{f} \right)$$





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# Sinusoids 2

**General expression for sinusoid**

$v(t) = V_m \sin(\omega t + \phi)$

$Argument \rightarrow \omega t + \phi$

$v_2 \; leads \; v_1 \; by \; \phi$

*Same (sin or cos) forms → in calculations*

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

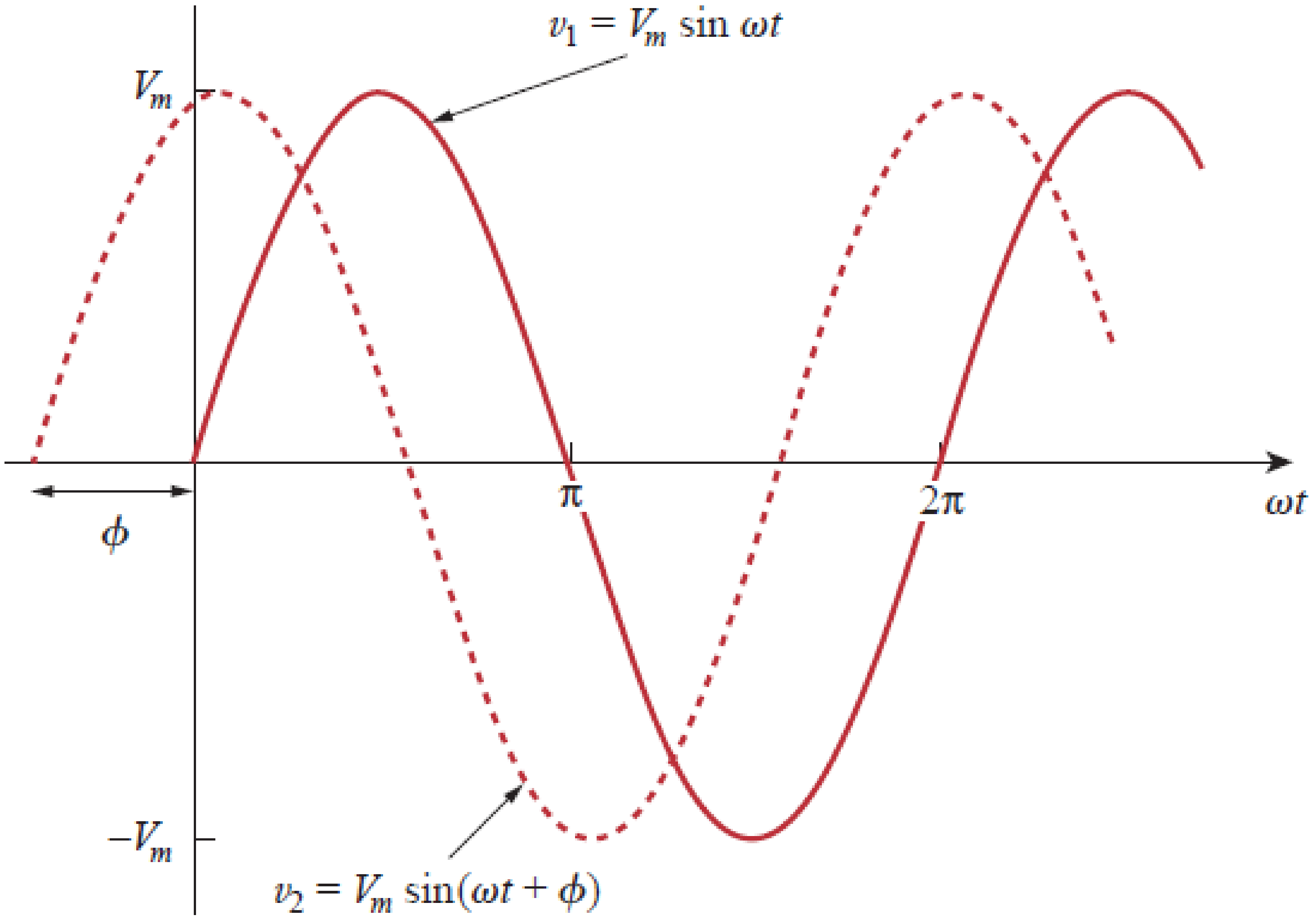
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin(\omega t \pm 180°) = -\sin\omega t$

$\cos(\omega t \pm 180°) = -\cos\omega t$

$\sin(\omega t \pm 90°) = \pm\cos\omega t$

$\cos(\omega t \pm 90°) = \mp\sin\omega t$





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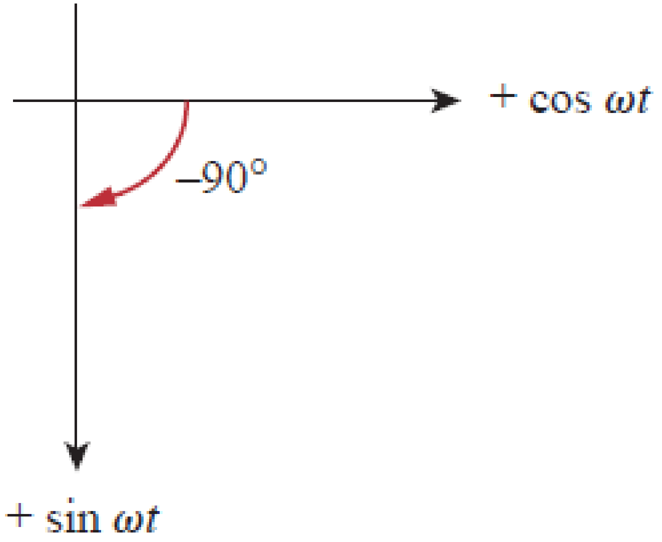
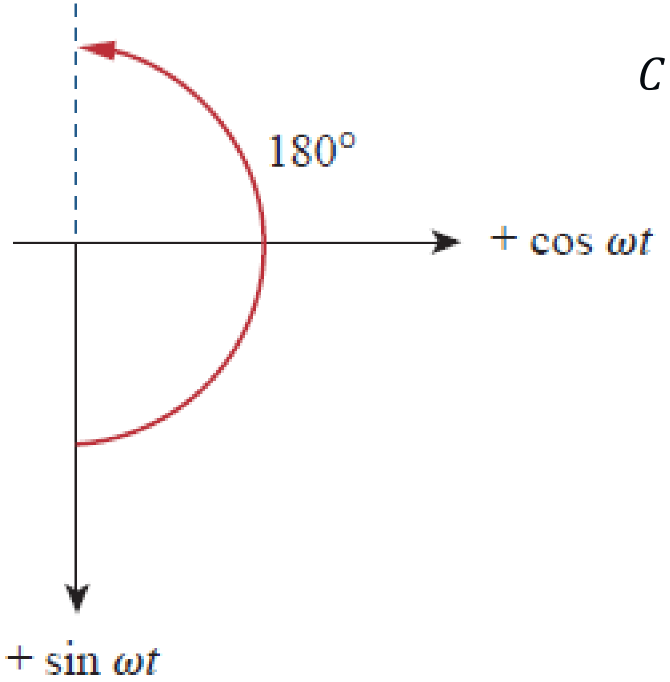
# Graphical Representation

$$\cos(\omega t \pm 90°) = \mp \sin \omega t$$

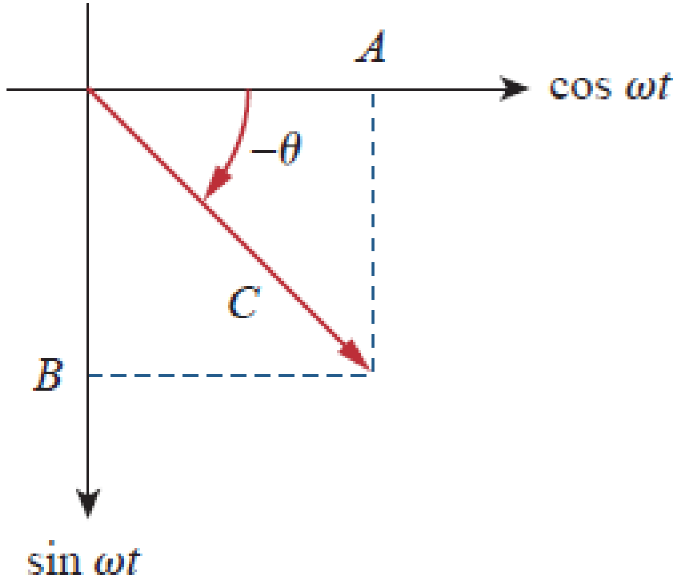
$$\sin(\omega t \pm 180°) = -\sin \omega t$$




$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \Theta)$$

$$C = \sqrt{A^2 + B^2}, \qquad \Theta = tan^{-1}\frac{B}{A}$$



# Phasors 1

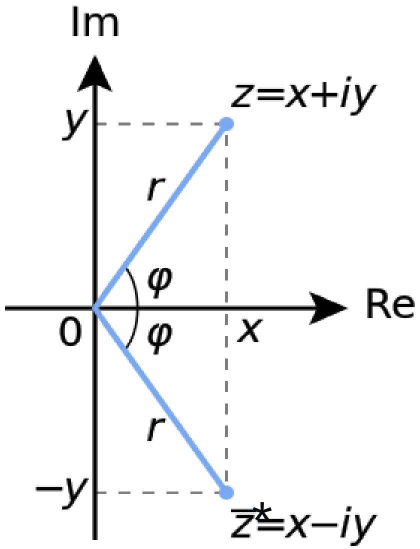
- Phasor → complex number … representing *amplitude* and *phase* of a sinusoid
- Suppressing time factor → transforming sinusoid from time domain to phasor domain.

$$rectangular \rightarrow x + j \cdot y$$
$$polar \rightarrow r \cdot e^{j\varphi}, (r/\varphi)$$

$j = \sqrt{-1}$

$$z = x + j \cdot y = r \cdot (\cos\varphi + j \cdot \sin\varphi) = r \cdot e^{j\varphi}$$

$$e^{j\varphi} = \cos\varphi + j \cdot \sin\varphi \leftarrow Euler's\ identity$$




$$r = \sqrt{x^2 + y^2}, \qquad \varphi = tan^{-1}\frac{y}{x}$$

$$x = r \cdot \cos\varphi, \qquad y = r \cdot \sin\varphi$$

$$\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}$$

$$\sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{j \cdot 2}$$

**Calculations**

$$z_1 = x_1 + j \cdot y_1, \qquad z_2 = x_2 + j \cdot y_2$$

$$z_1 + z_2 = (x_1 + x_2) + +j \cdot (y_1 + y_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{j(\varphi_1 + \varphi_2)}$$

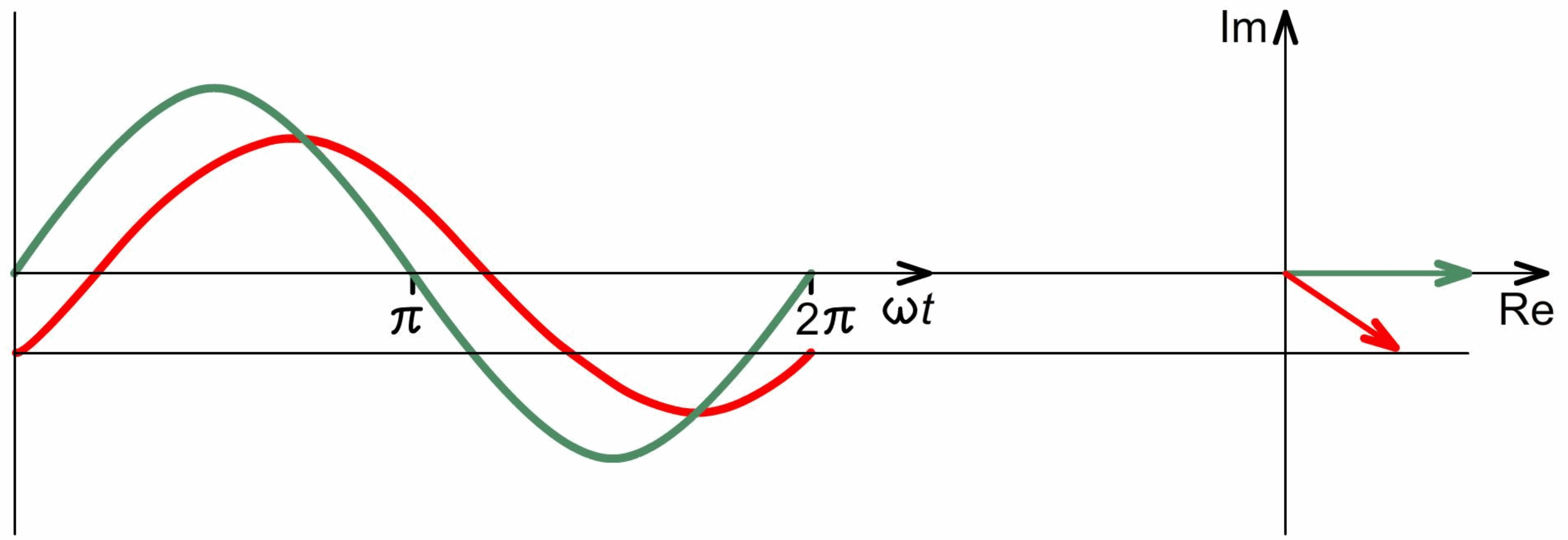
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{j(\varphi_1 - \varphi_2)}$$

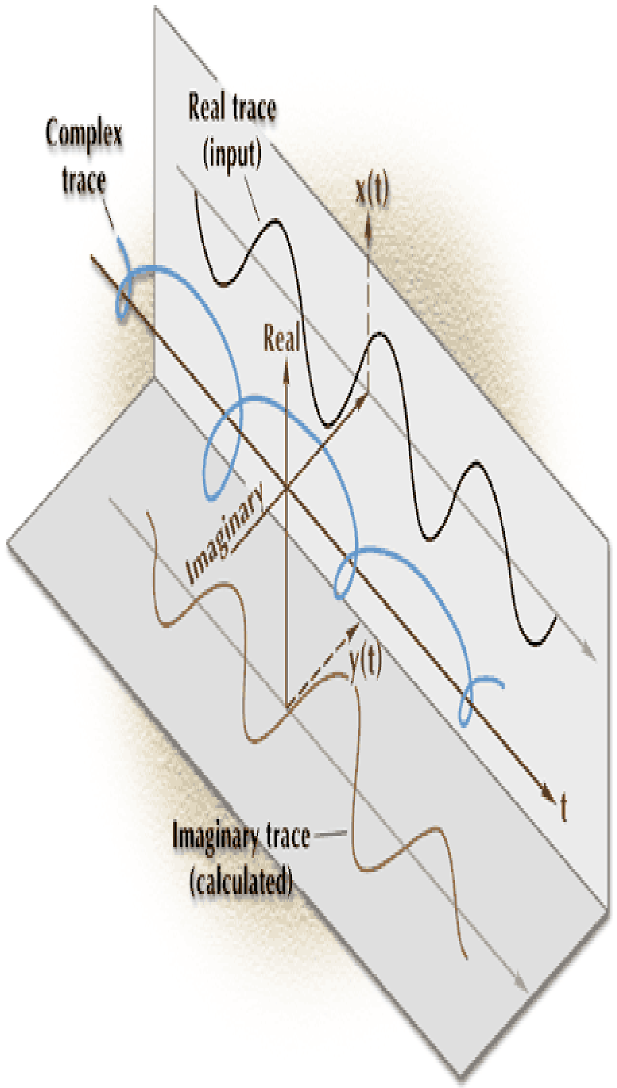
$$z^* = x - j \cdot y = r \cdot e^{-j\varphi}$$

# Phasors 2

$v(t) = V_m \cos(\omega\, t + \varphi)$ **transf.** $\boldsymbol{v}(t) = V_m[\cos(\omega\, t+\varphi) + j\sin(\omega\, t+\varphi)]$

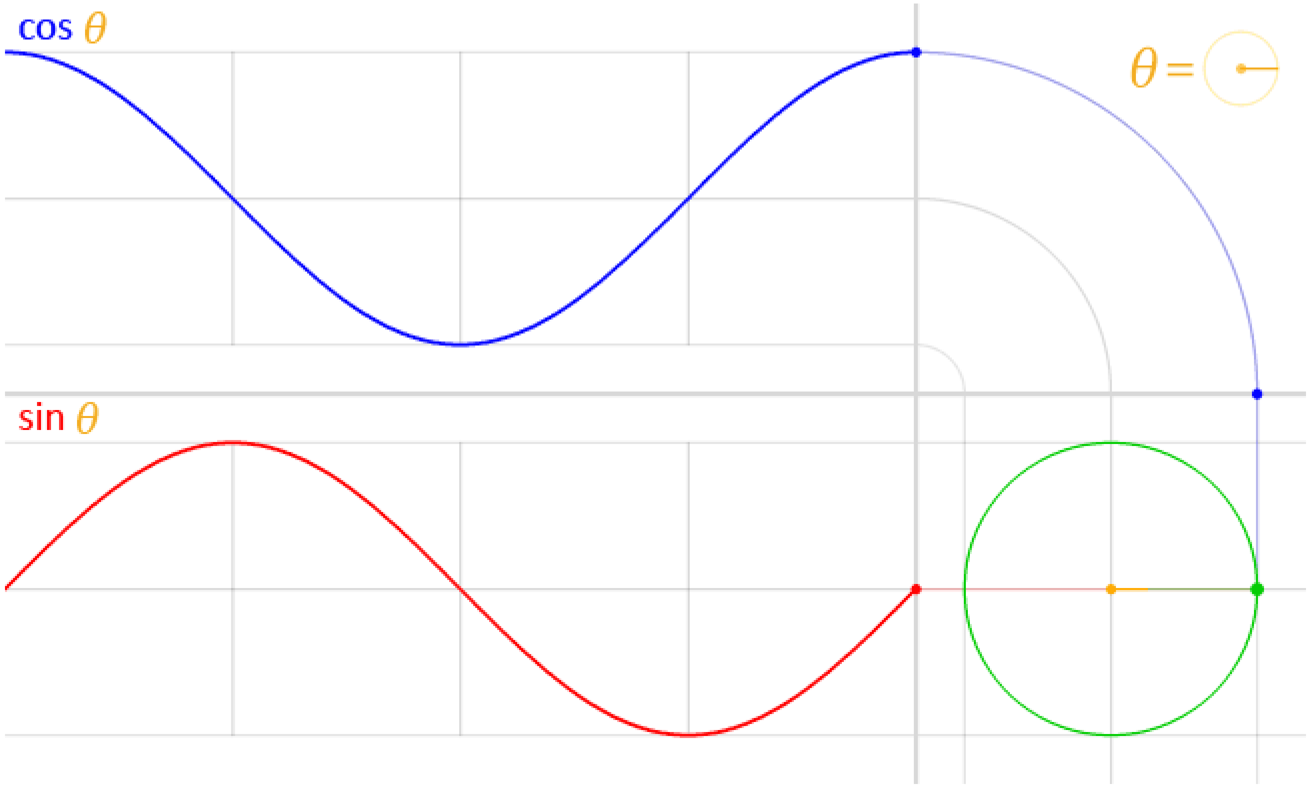
*(time dept. cpx. function)* $\boldsymbol{v}(t) = V_m\, e^{j(\omega\, t+\varphi)} = V_m\, e^{j\varphi}\, e^{j\omega\, t}$




$v(t) = Re\,\{\boldsymbol{v}(t)\}$ $PHASOR(supressing\ time\ factor) \rightarrow \boldsymbol{V}_m = V_m\, e^{j\varphi}$

# Phasor 3 (illustration)



# Transform time to phasor domain

$$v = V_m\,cos(\omega\,t+\varphi) \rightarrow \frac{dv}{dt} =? \rightarrow \frac{dv}{dt} = -\omega V_m\,sin(\omega\,t+\varphi)$$

$$= \omega V_m\,cos(\omega\,t+\varphi+90°) = Re\left\{\omega V_m e^{j\omega\,t} e^{j\varphi} e^{j90°}\right\}$$

$$= Re\left\{j\omega \boldsymbol{V} e^{j\omega\,t}\right\}$$

| Time domain | Phasor domain |
|---|---|
| $v(t) = V_m\,cos(\omega t+\varphi)$ | $\boldsymbol{V} = V_m\,e^{j\varphi}$ |
| $\frac{dv(t)}{dt}$ | $j\omega\,\boldsymbol{V}$ |
| $\int v(t)\cdot dt$ | $\frac{1}{j\omega}\boldsymbol{V}$ |

***COMMENTS***

- ❑ *Complex number → scalar!   (‚normal' z)*
- ❑ *Phasor → ‚vector-like' behavior (‚bold' **V**)*

**CONDITION!**

Phasor analysis → frequency is the same

**Properties of** *‚v(t)'*

- ❑ Instantaneous or time domain represent.
- ❑ Time dependent
- ❑ Always real

**Properties of ‚*V*'**

- ❑ Frequency or phasor domain represent.
- ❑ Time independent
- ❑ Generally complex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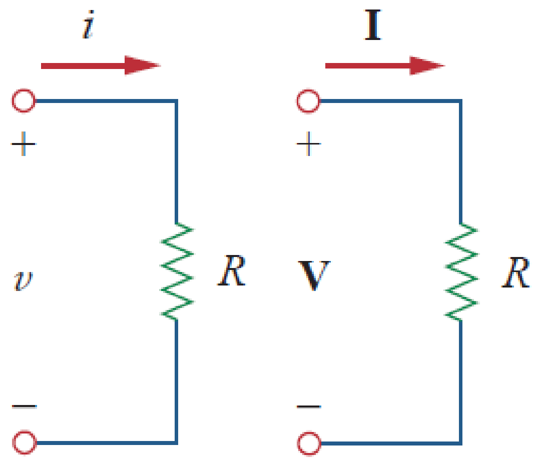
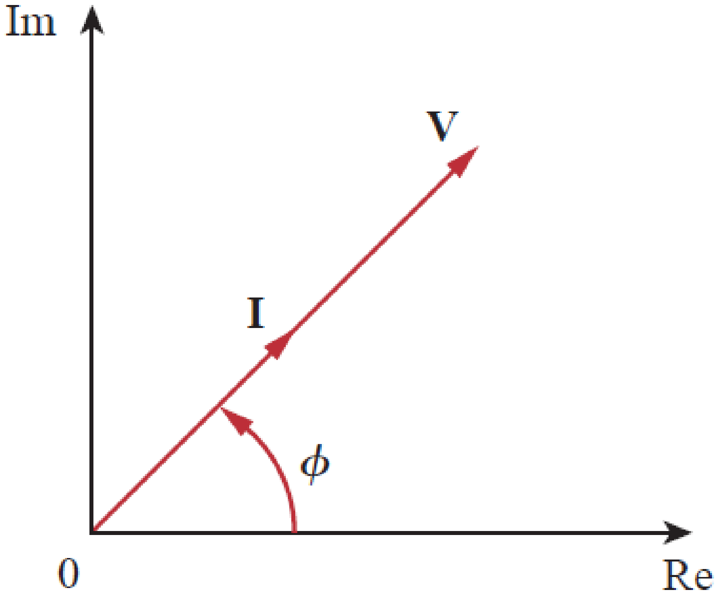
# Resistor in Phasor Domain




$$i = I_m \cdot \cos(\omega t + \varphi)$$

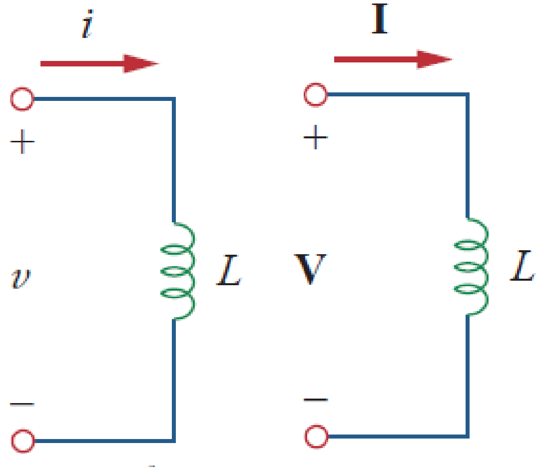
$$v = R \cdot i = R \cdot I_m \cdot \cos(\omega t + \varphi)$$

$$\boldsymbol{V} = R \cdot I_m \cdot e^{j\varphi}$$

$$\boldsymbol{I} = I_m \cdot e^{j\varphi} \rightarrow \boldsymbol{V} = R \cdot \boldsymbol{I}$$

# Inductor in Phasor Domain

,ELI'



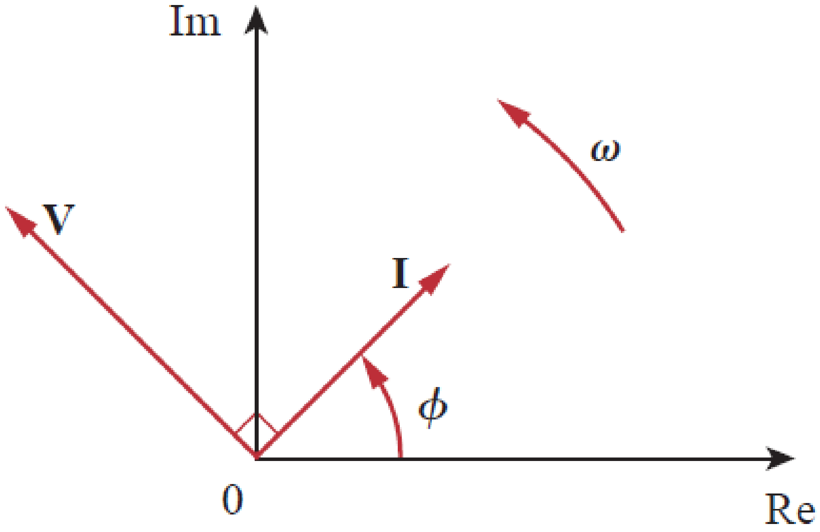

$$i = I_m \cdot \cos(\omega t + \varphi)$$

$$v = L \cdot \frac{di}{dt} = -\omega \cdot L \cdot I_m \cdot sin(\omega t + \varphi)$$

$$v = L \cdot \frac{di}{dt} = \omega \cdot L \cdot I_m \cdot cos(\omega t + \varphi + 90°)$$

$$\boldsymbol{V} = \omega \cdot L \cdot I_m \cdot e^{j(\varphi + 90°)} = \omega \cdot L \cdot I_m \cdot e^{j\varphi} \cdot e^{j90°}$$

$$\boldsymbol{I} = I_m \cdot e^{j\varphi}, e^{j90°} = j \rightarrow \boldsymbol{V} = \mathrm{j} \cdot \omega \cdot L \cdot \boldsymbol{I}$$





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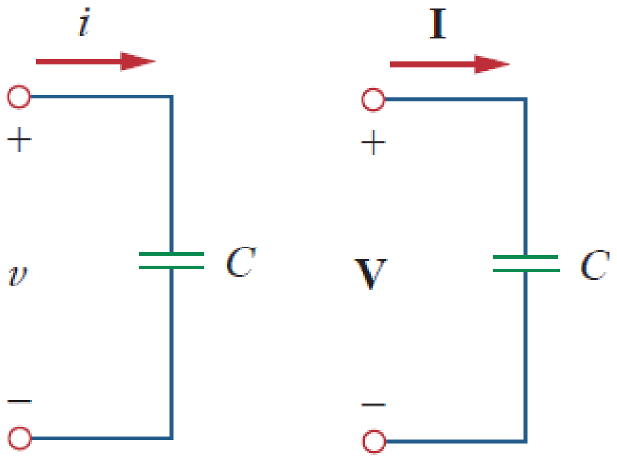
# Capacitor in Phasor Domain

‚ICE'



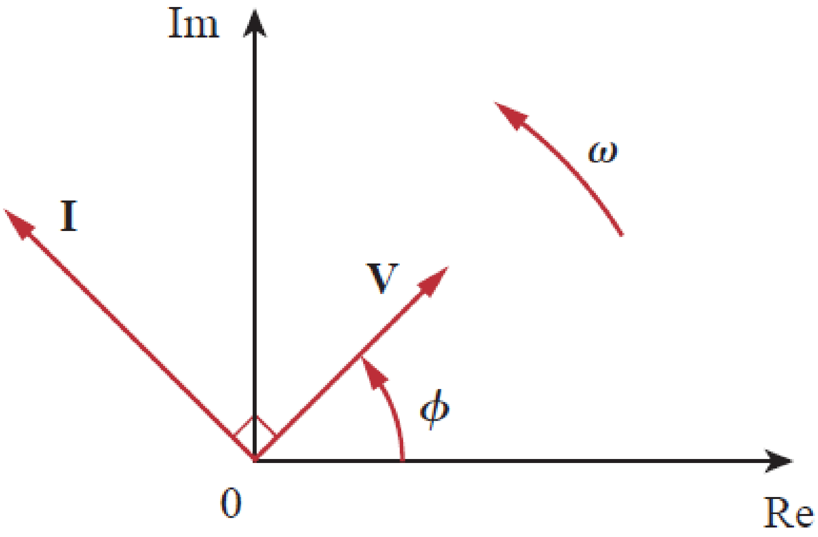

$$v = V_m \cdot \cos(\omega t + \varphi)$$

$$i = C\frac{dv}{dt} = -\omega \cdot C \cdot V_m \cdot \sin(\omega t + \varphi)$$

$$i = C \cdot \frac{dv}{dt} = \omega \cdot C \cdot V_m \cdot cos(\omega t + \varphi + 90°)$$

$$\boldsymbol{I} = \omega \cdot C \cdot V_m \cdot e^{j(\varphi + 90°)} = \omega \cdot C \cdot \boldsymbol{V}_m \cdot e^{j\varphi} \cdot e^{j90°}$$

$$\boldsymbol{I} = j \cdot \omega \cdot C \cdot \boldsymbol{V} = \rightarrow \boldsymbol{V} = \frac{1}{\mathrm{j} \cdot \omega \cdot C} \cdot \boldsymbol{I}$$





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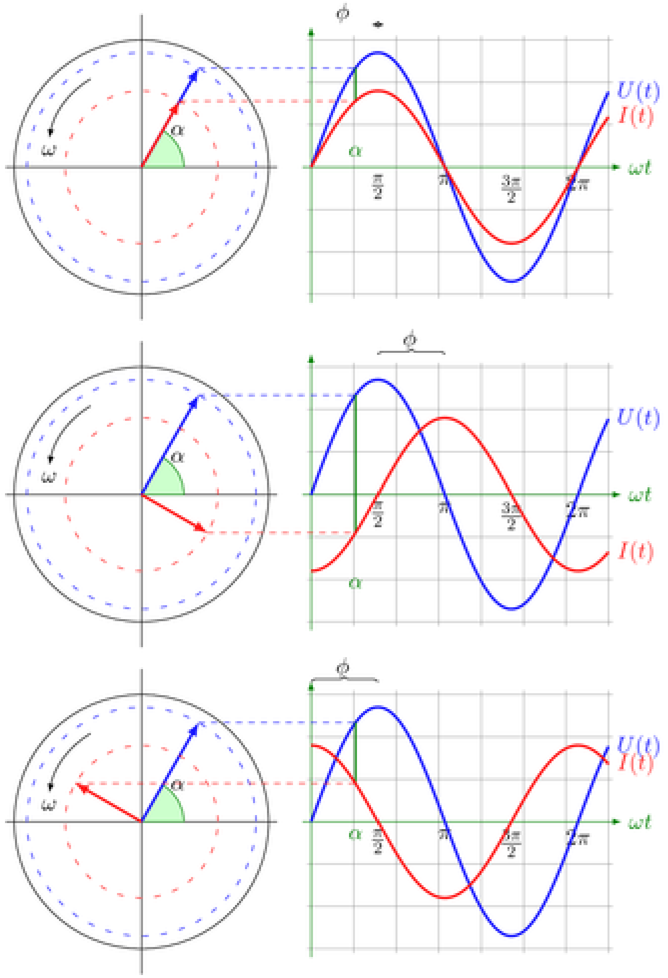
# Summary

| Element | Time Domain | Frequency Domain |
|---|---|---|
| $R$ | $v = R \cdot i$ | $\boldsymbol{V} = R \cdot \boldsymbol{I}$ |
| $L$ | $v = L \cdot \dfrac{di}{dt}$ | $\boldsymbol{V} = j \cdot \omega \cdot L \cdot \boldsymbol{I}$ |
| $C$ | $i = C \cdot \dfrac{dv}{dt}$ | $\boldsymbol{V} = \dfrac{1}{j \cdot \omega \cdot C} \cdot \boldsymbol{I}$ |





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# Impedance and Admittance

| Element | Time Domain | Frequency Domain |
|---|---|---|
| R | $v = R \cdot i$ | $\boldsymbol{V} = R \cdot \boldsymbol{I}$ |
| L | $v = L \cdot \dfrac{di}{dt}$ | $\boldsymbol{V} = j\omega L \cdot \boldsymbol{I}$ |
| C | $i = C \cdot \dfrac{dv}{dt}$ | $\boldsymbol{V} = \dfrac{1}{j\omega C} \cdot \boldsymbol{I}$ |

$$Ohm's\ law\ in\ phasor\ form \rightarrow \boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}}, \qquad \boldsymbol{Y} = \frac{\boldsymbol{I}}{\boldsymbol{V}}$$

$$\boldsymbol{Z} = R + j \cdot X, \qquad \boldsymbol{Y} = G + j \cdot B$$

**Impedance**

- Denoted by the letter **Z** (complex)
- Ratio of the phasor $\boldsymbol{V}$ to the phasor $\boldsymbol{I}$
- Measured in ohms ($\Omega$)

**Admittance**

- Reciprocal of impedance
- Denoted by the letter **Y** (complex)
- Measured in siemens (S).

- *R*: resistance (real part of the impedance)
- *X*: reactance (imaginary part of the impedance)
- *G*: conductance (real part of the admittance)
- *B*: susceptance (imaginary part of the admittance)

$$\boldsymbol{Z}_R = R, \quad \boldsymbol{Z}_L = \mathrm{j}\omega L = jX_L, \quad \boldsymbol{Z}_C = \frac{1}{j\omega C} = -jX_C, \quad \boldsymbol{Y}_R = G, \quad \boldsymbol{Y}_L = \frac{1}{j\omega L} = -jB_L, \quad \boldsymbol{Y}_C = j\omega C = jB_C$$


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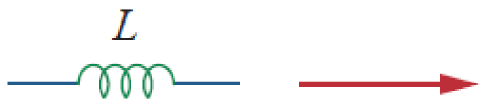
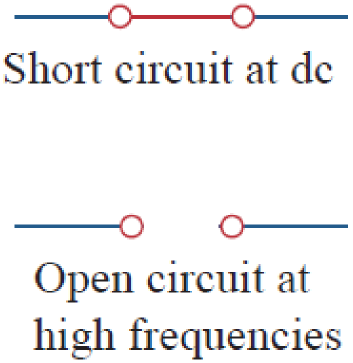
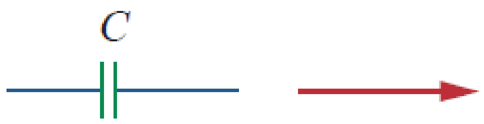
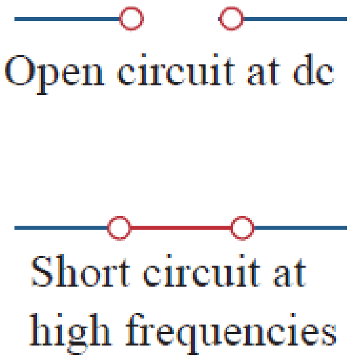
# Impedance and Admittance




$$\mathbf{Z}_L = jX_L = j\omega L$$




$$\mathbf{Z}_C = \frac{1}{j\omega C} = -jX_C$$

$$\mathbf{Z} = R + jX \rightarrow \begin{cases} Z = |\mathbf{Z}| = \sqrt{R^2 + X^2} \\ \Theta = tan^{-1}\dfrac{X}{R} \end{cases} \qquad \begin{aligned} R &= Z\cos\Theta \\ X &= Z\sin\Theta \end{aligned}$$

$$\mathbf{Y} = G + jB \rightarrow \begin{cases} Y = |\mathbf{Y}| = \sqrt{G^2 + B^2} \\ \Theta = tan^{-1}\dfrac{B}{G} \end{cases} \qquad \begin{aligned} G &= Y\cos\Theta \\ B &= Y\sin\Theta \end{aligned}$$

$$G + jB = \frac{1}{R + jX} \quad \rightarrow G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2}$$

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2}$$

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# Kirchhoff’s Laws in Freq. Domain

- **KCL** → $i_1 + i_2 + i_3 + \cdots + i_n = 0$

$$I_{m1} \cdot cos(\omega t + \varphi_1) + I_{m2} \cdot cos(\omega t + \varphi_2) + I_{m3} \cdot cos(\omega t + \varphi_3) + \cdots + I_{mn} \cdot cos(\omega t + \varphi_n) = 0$$

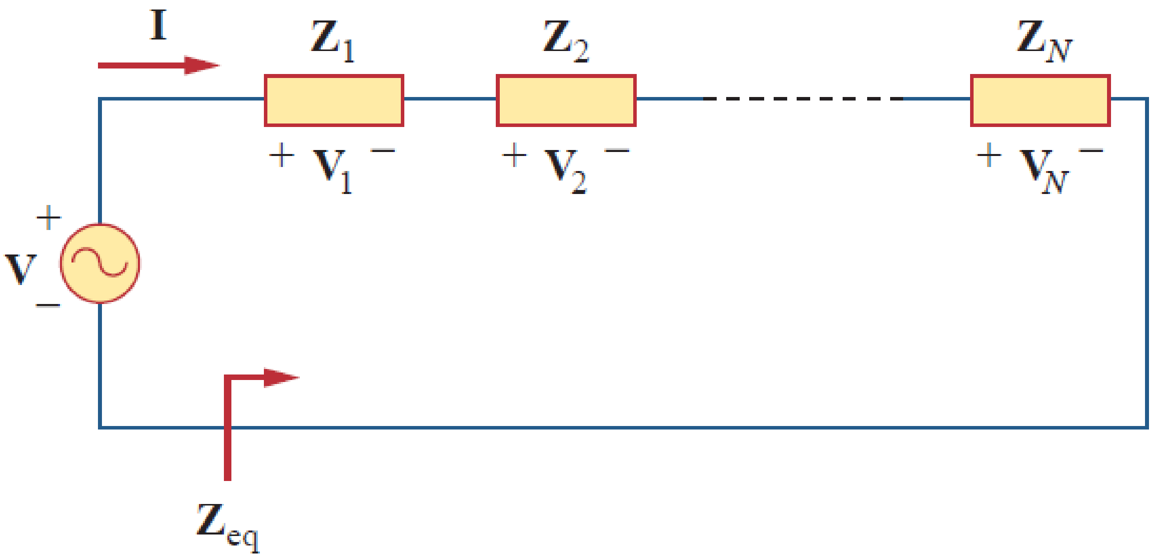
$$Re\{I_{m1} \cdot e^{j\varphi_1} \cdot e^{j\omega t}\} + Re\{I_{m2} \cdot e^{j\varphi_2} \cdot e^{j\omega t}\} + Re\{I_{m3} \cdot e^{j\varphi_3} \cdot e^{j\omega t}\} + \cdots + Re\{I_{mn} \cdot e^{j\varphi_n} \cdot e^{j\omega t}\} = 0$$

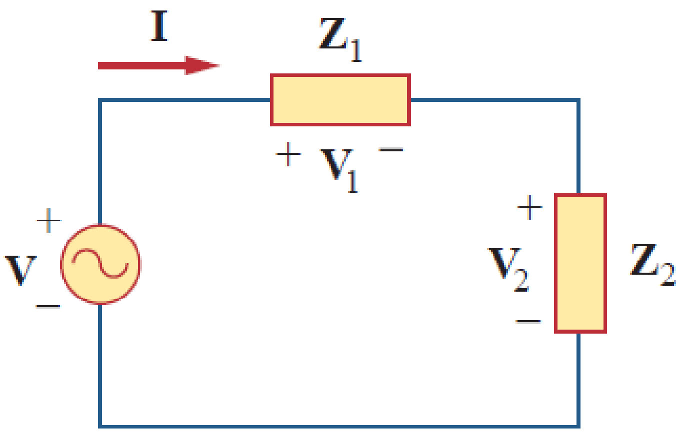
$$\boldsymbol{I}_k = I_{mk} \cdot e^{j\varphi_k} \rightarrow Re\{(\boldsymbol{I}_1 + \boldsymbol{I}_2 + \boldsymbol{I}_3 + \cdots + \boldsymbol{I}_n) \cdot e^{j\omega t}\} = 0$$

$$e^{j\omega t} \neq 0 \rightarrow \boldsymbol{I}_1 + \boldsymbol{I}_2 + \boldsymbol{I}_3 + \cdots + \boldsymbol{I}_n = 0$$

- **KVL** → $v_1 + v_2 + v_3 + \cdots + v_n = 0 \rightarrow \cdots \rightarrow \boldsymbol{V}_1 + \boldsymbol{V}_2 + \boldsymbol{V}_3 + \cdots + \boldsymbol{V}_n = 0$

# Series Impedances, Voltage Division



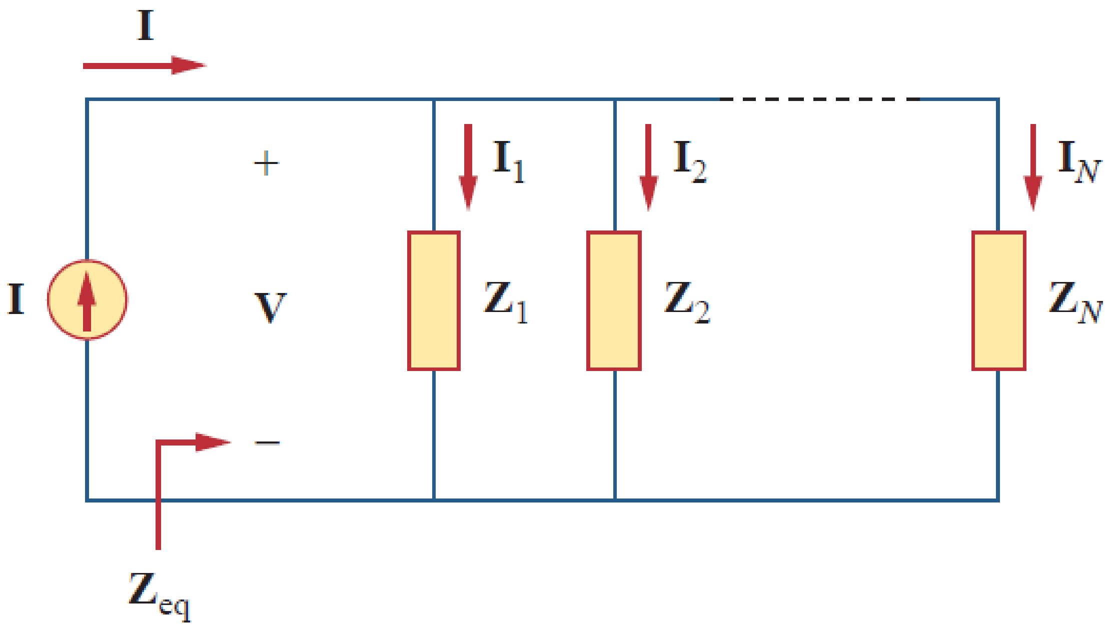

$$V = V_1 + V_2 + V_3 + \cdots + V_N = I(Z_1 + Z_2 + Z_3 + \cdots + Z_N)$$

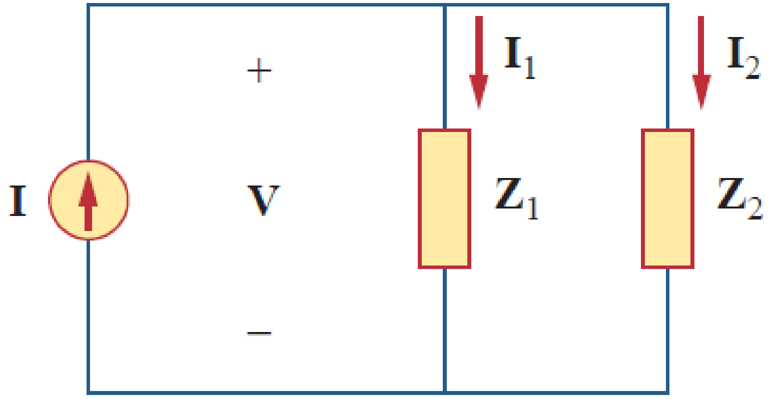
$$I = \frac{V}{Z_1 + Z_2}, V_1 = IZ_1, V_2 = IZ_2$$

$$Z_{eq} = \frac{V}{I} = Z_1 + Z_2 + Z_3 + \cdots + Z_N \rightarrow Z_{eq} = \sum_{i=1}^{N} Z_i$$

$$V_1 = V\frac{Z_1}{Z_1 + Z_2}, V_2 = V\frac{Z_2}{Z_1 + Z_2}$$

# Parallel Impedances, Current Division



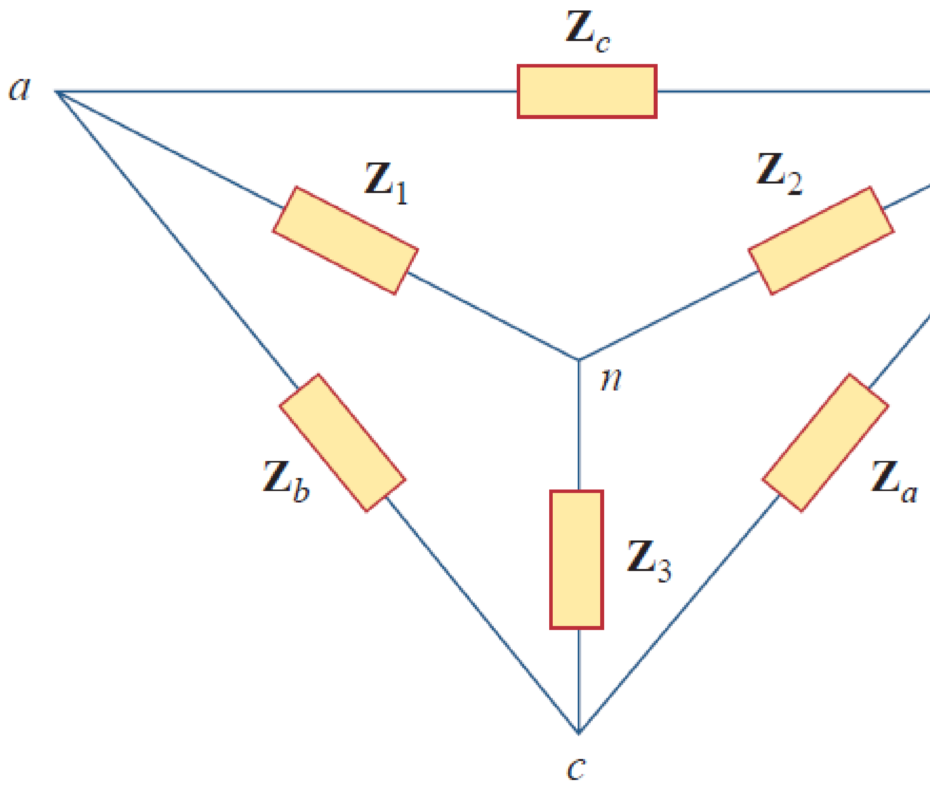

$$I = I_1 + I_2 + \cdots + I_N = V\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}\right)$$

$$Z_{eq} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}\,, \qquad V = IZ_{eq} = I_1 Z_1 = I_2 Z_2$$

$$\frac{1}{Z_{eq}} = \frac{I}{V} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N} \rightarrow Y_{eq} = \sum_{i=1}^{N} Y_i$$

$$I_1 = I\frac{Z_2}{Z_1 + Z_2}\,, \qquad I_2 = I\frac{Z_1}{Z_1 + Z_2}$$

# Wye – Delta Transform




*(like it is in DC circuits…)*

$$\boldsymbol{Z}_1 = \frac{\boldsymbol{Z}_b \boldsymbol{Z}_c}{\boldsymbol{Z}_a + \boldsymbol{Z}_b + \boldsymbol{Z}_c} \qquad \boldsymbol{Z}_a = \frac{\boldsymbol{Z}_1\boldsymbol{Z}_2 + \boldsymbol{Z}_2\boldsymbol{Z}_3 + \boldsymbol{Z}_1\boldsymbol{Z}_3}{\boldsymbol{Z}_1}$$

$$\boldsymbol{Z}_2 = \frac{\boldsymbol{Z}_a \boldsymbol{Z}_c}{\boldsymbol{Z}_a + \boldsymbol{Z}_b + \boldsymbol{Z}_c} \qquad \boldsymbol{Z}_b = \frac{\boldsymbol{Z}_1\boldsymbol{Z}_2 + \boldsymbol{Z}_2\boldsymbol{Z}_3 + \boldsymbol{Z}_1\boldsymbol{Z}_3}{\boldsymbol{Z}_2}$$

$$\boldsymbol{Z}_3 = \frac{\boldsymbol{Z}_a \boldsymbol{Z}_b}{\boldsymbol{Z}_a + \boldsymbol{Z}_b + \boldsymbol{Z}_c} \qquad \boldsymbol{Z}_c = \frac{\boldsymbol{Z}_1\boldsymbol{Z}_2 + \boldsymbol{Z}_2\boldsymbol{Z}_3 + \boldsymbol{Z}_1\boldsymbol{Z}_3}{\boldsymbol{Z}_3}$$

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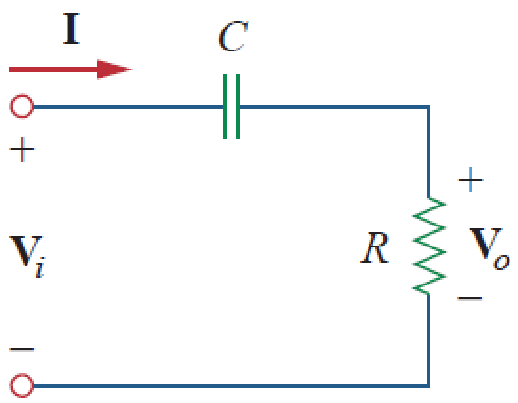
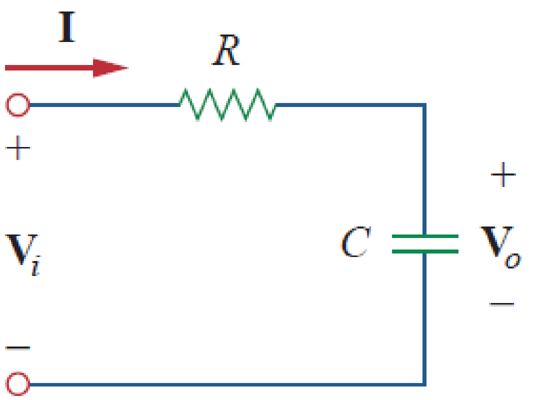
- ❑ **Application Example – Phase Shifters**

# App. – Phase Shifters



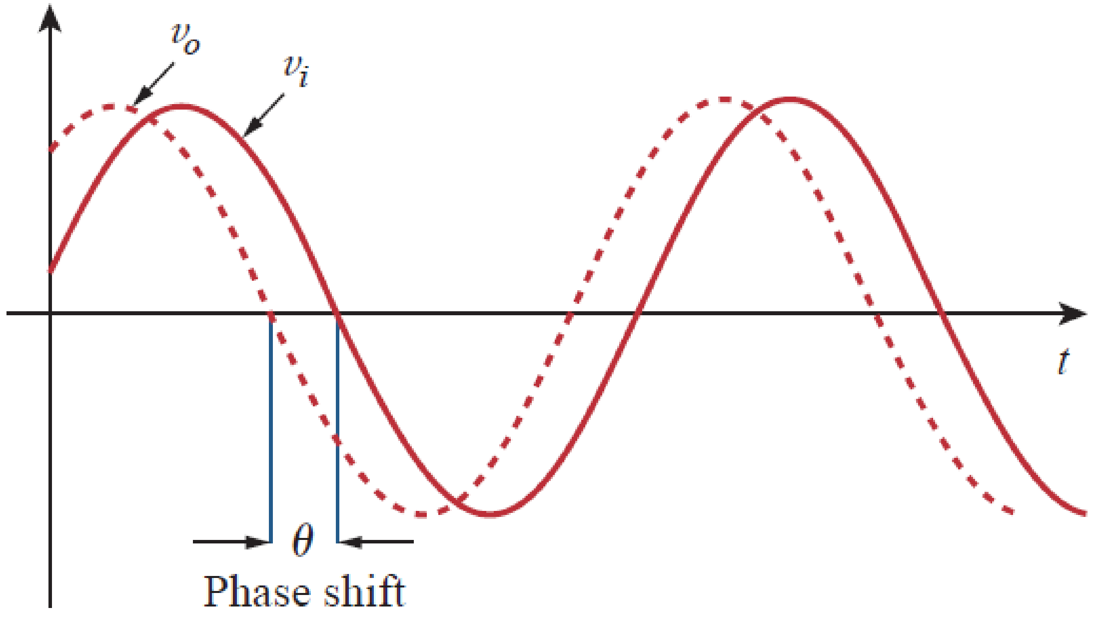

$$\frac{V_o}{V_i}=\frac{R}{R-jX_C}=\frac{R(R+jX_C)}{R^2+X_C^2}$$

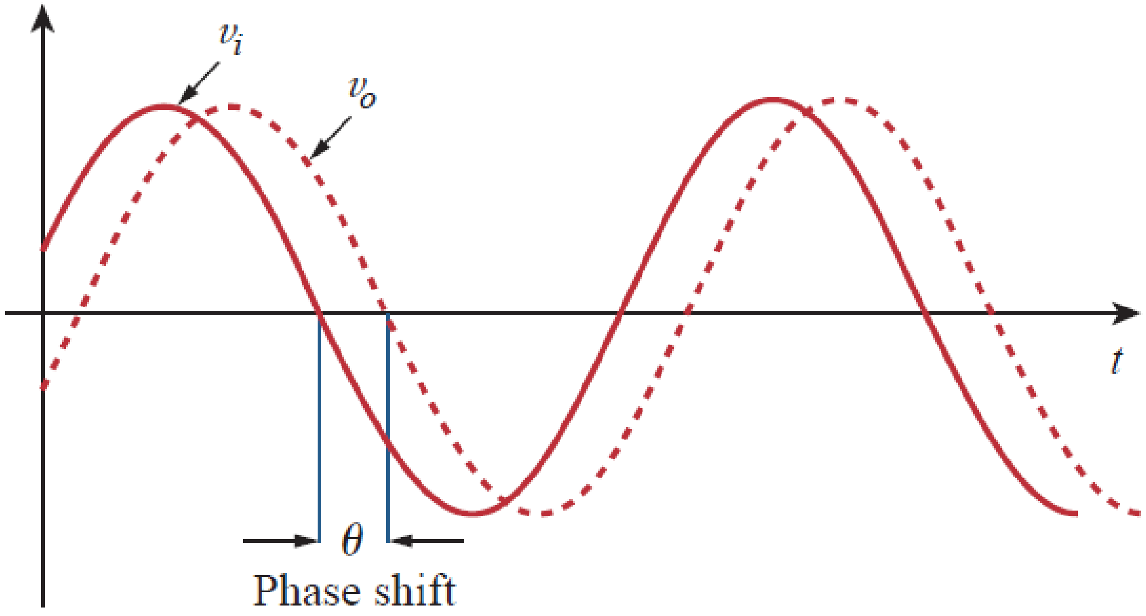
$$\Theta = tan^{-1}\frac{X_C}{R} \leftarrow (0 \ldots 90°)$$




$$\frac{V_o}{V_i}=\frac{-jX_C}{R-jX_C}=\frac{X_C(X_C-jR)}{R^2+X_C^2}$$

$$\Theta = tan^{-1}\left(-\frac{R}{X_C}\right) \leftarrow (-90° \ldots 0)$$





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# Questions