DR. GYURCSEK ISTVÁN

Magnetically Coupled Circuits

Sources and additional materials (recommended)

- *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- *W. M. Flanagan, Handbook of Transformer Design and Applications, 2nd ed. (New York: McGraw-Hill, 1993)*
- *Mayergoyz - Lawson: Basic Electric Circuit Theory (ISBN13: 978-0124808652)*
- *Simonyi K.: Villamosságtan. AK Budapest 1983, ISBN:9630534134*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*
- *Zombory L.: Elektromágneses terek. MK Budapest 2006, (www.electro.uni-miskolc.hu)*

- **Coupled Inductances**
- Linear Transformers
- Ideal Transformers
- Three-Phase Transformers
- Transformer Applications

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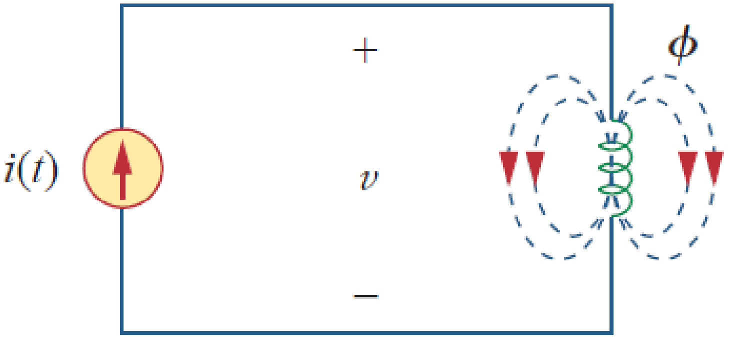
Self-Inductance *(Recall)*

$$v = N\frac{d\Phi}{dt} = N\frac{d\Phi}{di}\frac{di}{dt} = L\frac{di}{dt}$$

Self-inductance $L = N\frac{d\Phi}{di}$



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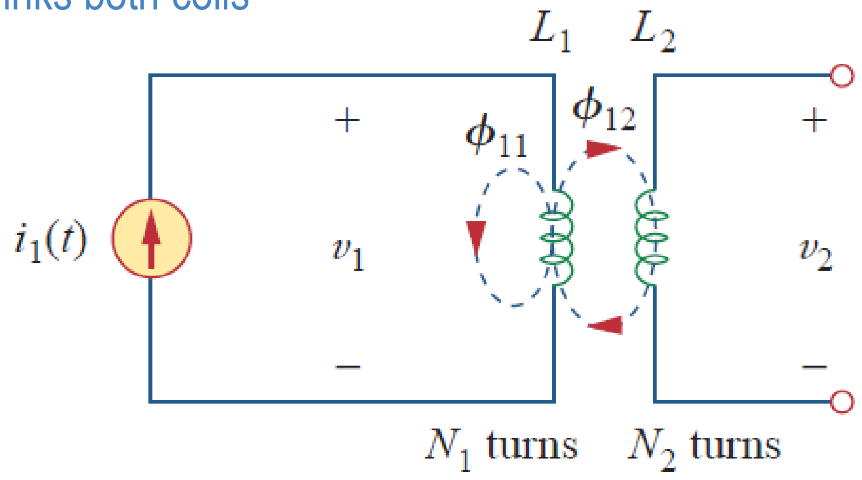
Mutual Inductance *(Recall)*

$\Phi_1 = \Phi_{11} + \Phi_{12}$ $\mathbf{\Phi}_{11}$: links only coil 1, $\mathbf{\Phi}_{12}$: links both coils

$$v_1 = N_1 \frac{d\Phi_1}{dt} = N_1 \frac{d\Phi_1}{di_1}\frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

$$v_2 = N_2 \frac{d\Phi_{12}}{dt} = N_2 \frac{d\Phi_{12}}{di_1}\frac{di_1}{dt} = M_{21} \frac{di_1}{dt}$$

Mutual inductance $M_{21} = N_2 \frac{d\Phi_{12}}{di_1}$



OC mutual *(induced)* voltage

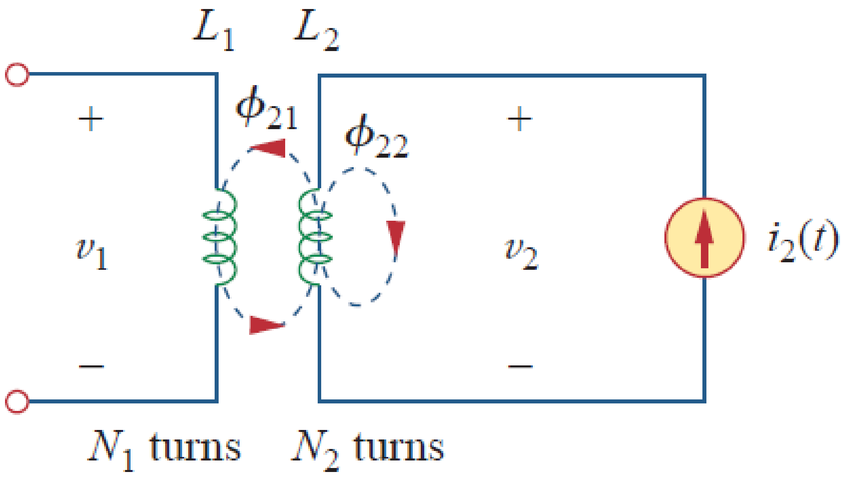
$$v_2 = M_{21} \frac{di_1}{dt}$$

$\Phi_2 = \Phi_{21} + \Phi_{22}$

$$v_2 = N_2 \frac{d\Phi_2}{dt} = N_2 \frac{d\Phi_2}{di_2}\frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

$$v_1 = N_1 \frac{d\Phi_{21}}{dt} = N_1 \frac{d\Phi_{21}}{di_2}\frac{di_2}{dt} = M_{12} \frac{di_2}{dt}$$

Mutual inductance $M_{12} = N_1 \frac{d\Phi_{21}}{di_2}$



OC mutual *(induced)* voltage

$$v_1 = M_{12} \frac{di_2}{dt}$$

$(M_{12} = M_{21} = M)$

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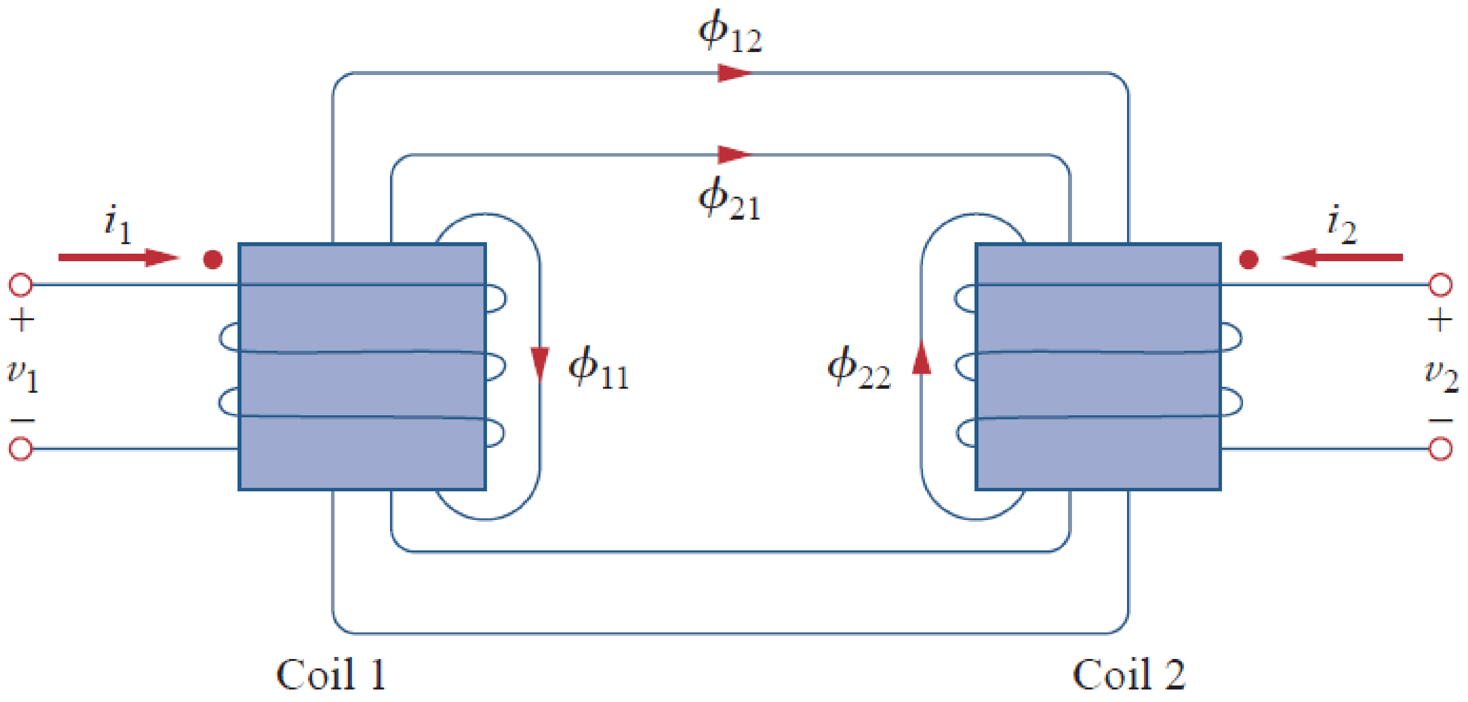
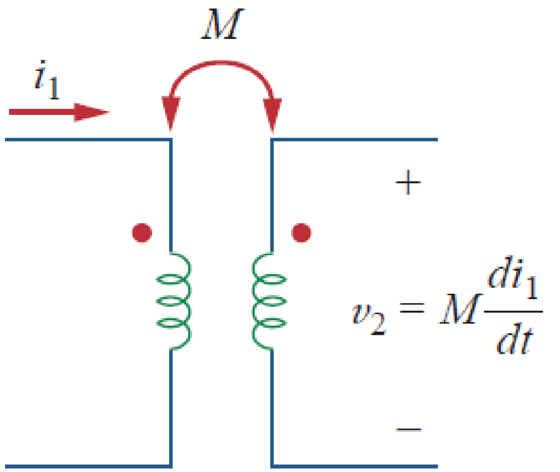
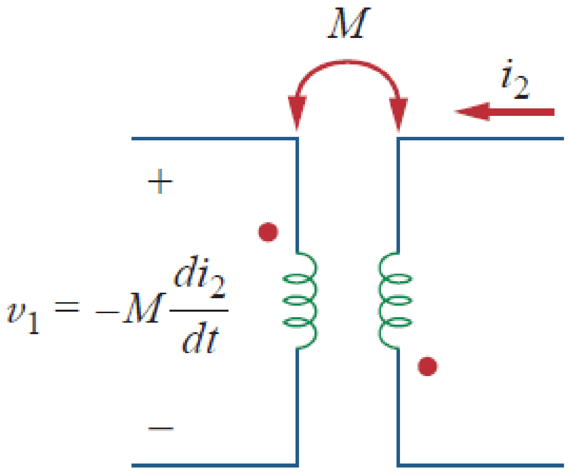
Dot Convention

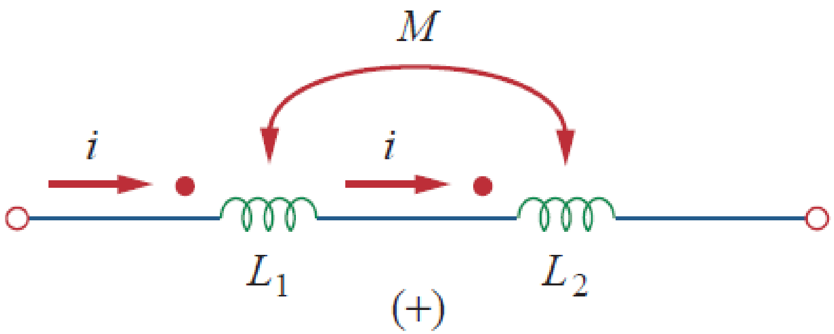


[current enters at dot] → [mutual voltage positive at dot]

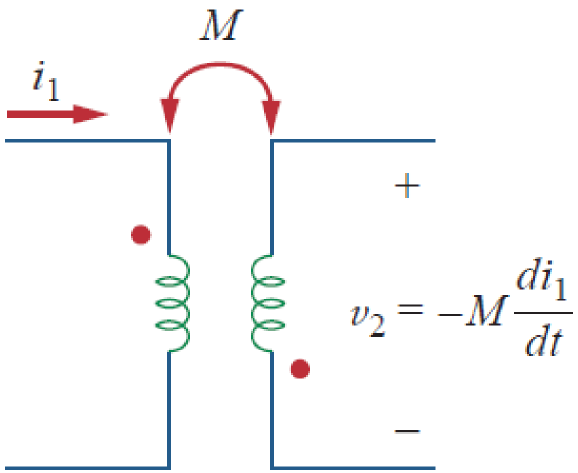
Dot Convention - Examples

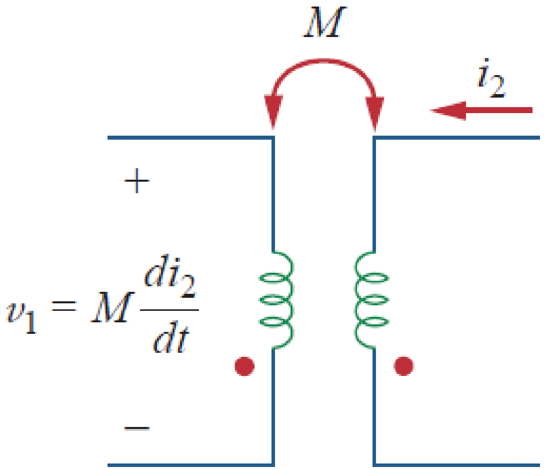


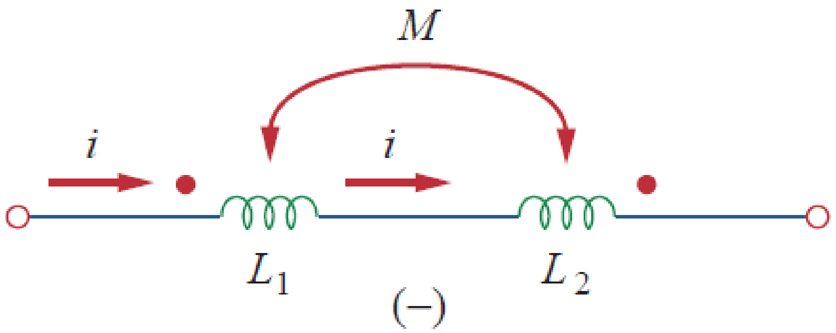




Series-aiding connection → $L = L_1 + L_2 + 2M$







Series-oppositing connection → $L = L_1 + L_2 - 2M$

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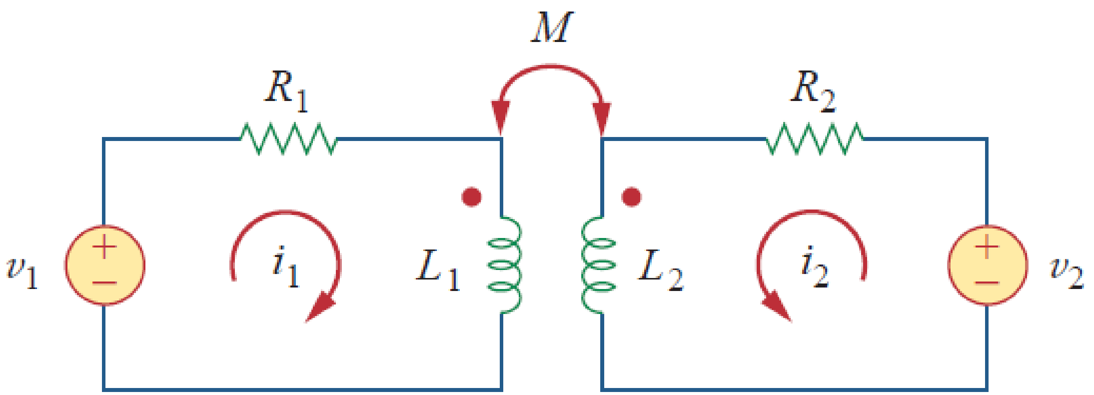
Time and Frequency Domain Analysis

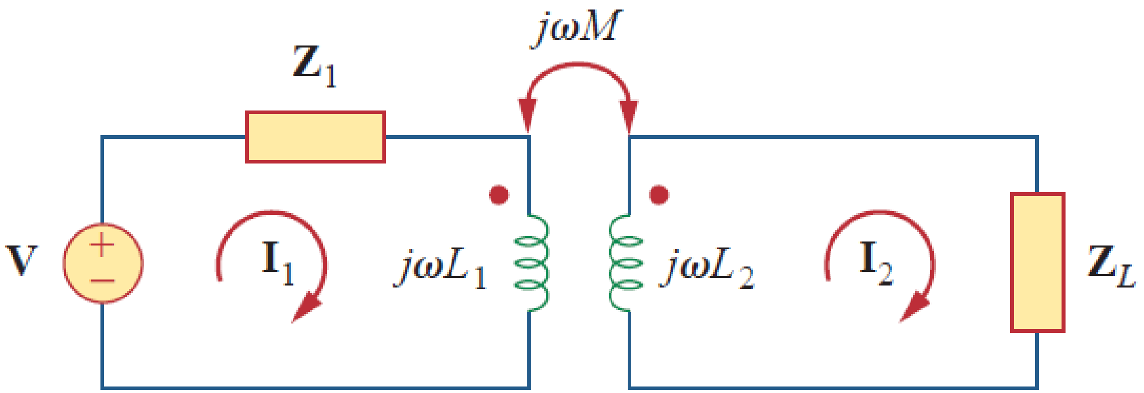


$$v_1 = i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = i_2 R_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$\boldsymbol{V}_1 = (R_1 + j\omega L_1)\boldsymbol{I}_1 + j\omega M \boldsymbol{I}_2$$

$$\boldsymbol{V}_2 = j\omega M \boldsymbol{I}_1 + (R_2 + j\omega L_2)\boldsymbol{I}_2$$



$$\boldsymbol{V} = (\boldsymbol{Z}_1 + j\omega L_1)\boldsymbol{I}_1 - j\omega M \boldsymbol{I}_2$$

$$0 = -j\omega M \boldsymbol{I}_1 + (\boldsymbol{Z}_L + j\omega L_2)\boldsymbol{I}_2$$

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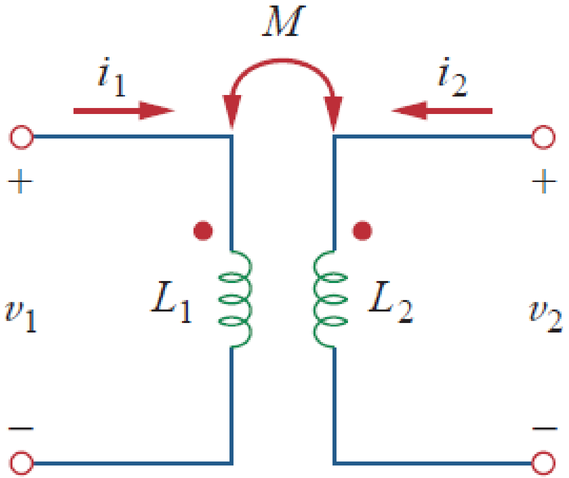
Energy in Coupled Circuit

Approaching in two steps...

$$(\mathbf{1}):\ i_1 = (0 \to I_1), i_2 = 0 \to p_1(t) = v_1 i_1 = i_1 L_1 \frac{di_1}{dt}$$

$$\to w_1 = \int p_1\, dt = L_1 \int_0^{I_1} i_1\, di_1 = \frac{1}{2} L_1 I_1^2$$



$$(\mathbf{2}):\ i_1 = I_1, i_2 = (0 \to I_2) \to p_2(t) = i_1 M \frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt}$$

$$\to w_2 = \int p_2\, dt = M I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2 di_2 = M I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

Total w (both of currents → max value) $w = w_1 + w_2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M I_1 I_2$ ... ($\pm \leftarrow$ dot convention!)

Coupling Coefficient

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$$

$$\text{Passive circuit} \rightarrow \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 - M i_1 i_2 \geq 0$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\rightarrow \frac{1}{2}\left(i_1\sqrt{L_1} - i_2\sqrt{L_2}\right)^2 + i_1 i_2\left(\sqrt{L_1 L_2} - M\right) \geq 0$$

$$\rightarrow \sqrt{L_1 L_2} - M \geq 0$$

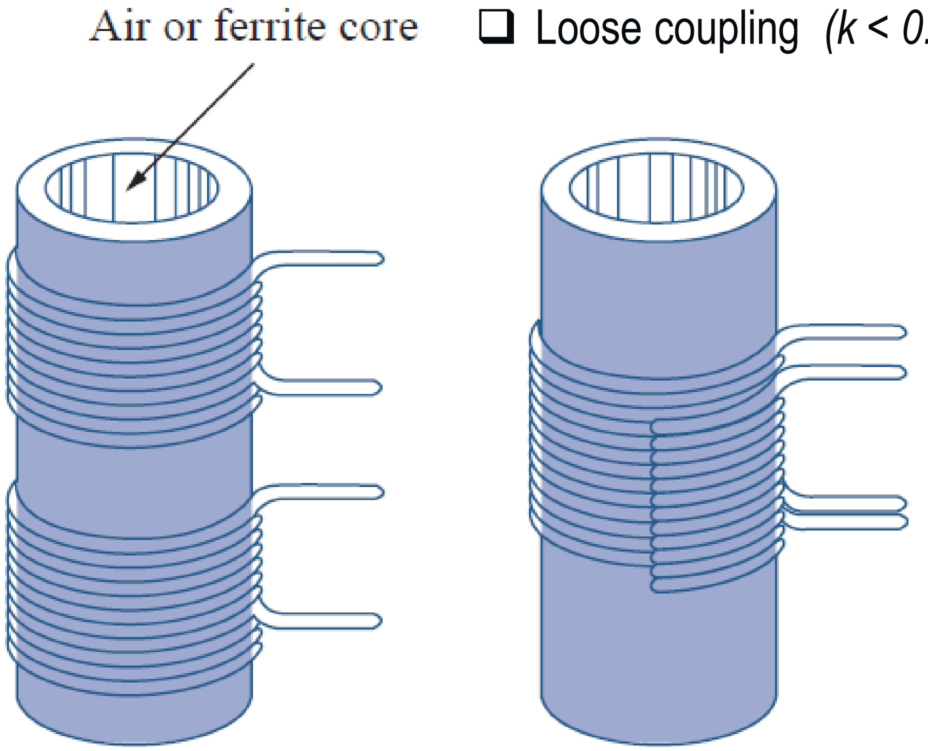
$$M \leq \sqrt{L_1 L_2} \rightarrow M = k\sqrt{L_1 L_2}$$

k → measure of the magn. coupling bw. coils (0 - 1)

$$k = \frac{\Phi_{12}}{\Phi_1} = \frac{\Phi_{12}}{\Phi_{12} + \Phi_{11}} = \frac{\Phi_{21}}{\Phi_2} = \frac{\Phi_{21}}{\Phi_{21} + \Phi_{22}}$$

Coupling coefficient

- ❑ Perfekt coupling... *(k = 1)*
- ❑ Tight coupling *(k > 0.5)*
- ❑ Loose coupling *(k < 0.5)*



- Coupled Inductances
- **Linear Transformers**
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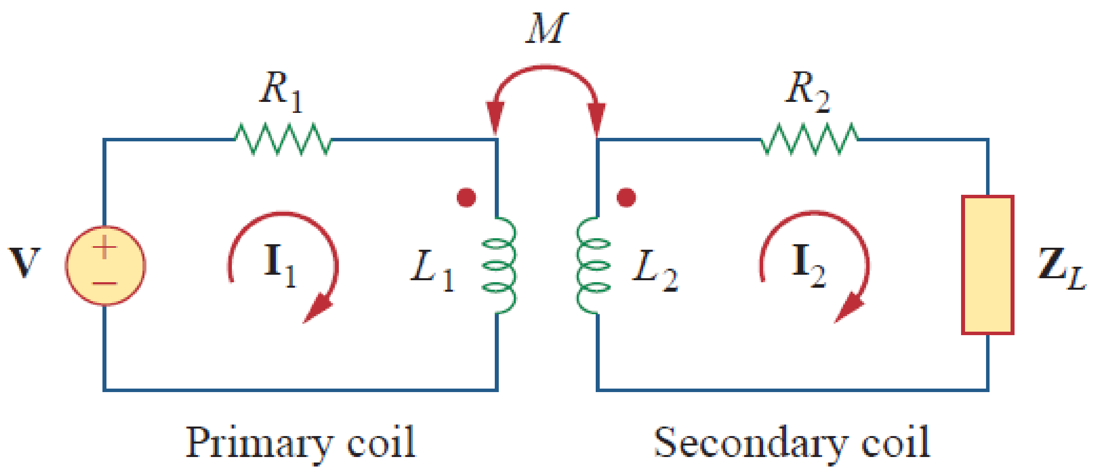
Linear Transformers

Linear transformer → $\Phi \sim I$



$$\boldsymbol{V} = (R_1 + j\omega L_1)\boldsymbol{I}_1 - j\omega M \boldsymbol{I}_2$$

$$0 = -j\omega M \boldsymbol{I}_1 + (R_2 + j\omega L_2 + \boldsymbol{Z}_L)\boldsymbol{I}_2$$

Primary impedance $\boldsymbol{Z}_{PR} = R_1 + j\omega L_1$

$$\boldsymbol{Z}_{in} = \frac{\boldsymbol{V}}{\boldsymbol{I}_1} = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \boldsymbol{Z}_L}$$

Reflected (coupled) impedance $\boldsymbol{Z}_R = \dfrac{\omega^2 M^2}{R_2 + j\omega L_2 + \boldsymbol{Z}_L}$

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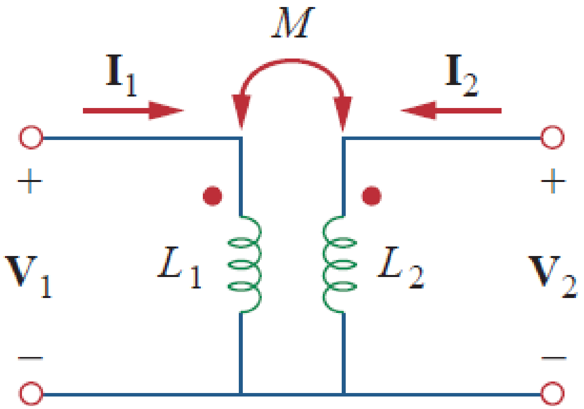
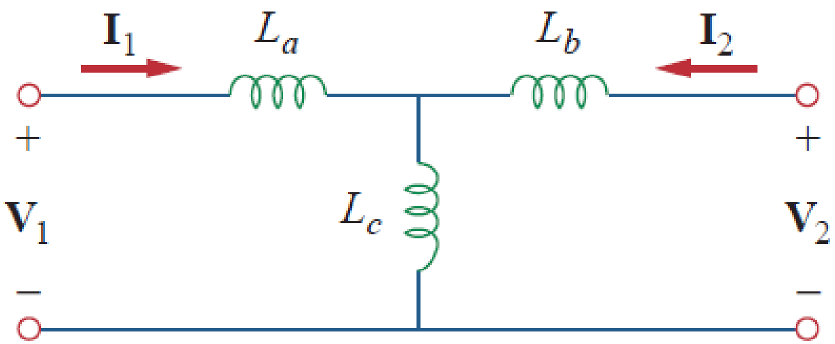
Equivalent T - Circuit

Equivalent circuits → convenient analysis



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$



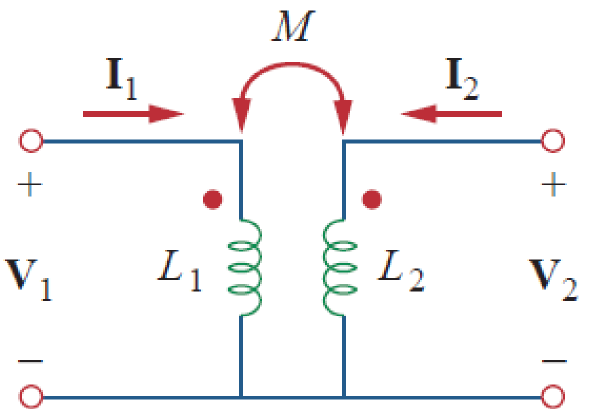
mesh analysis → $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_c) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$

$L_a = L_1 - M$, $L_b = L_2 - M$, $L_C = M$

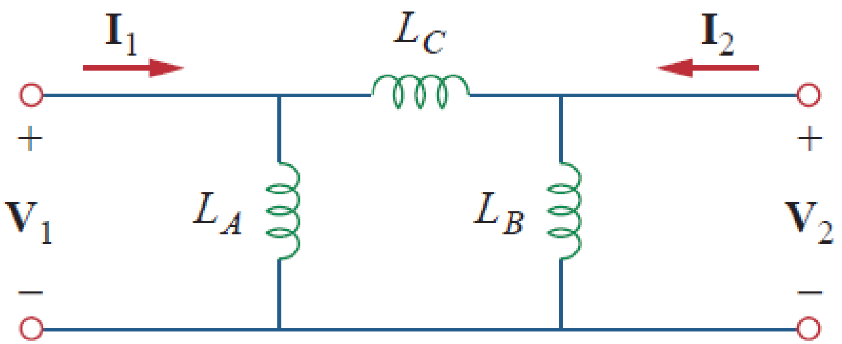
Equivalent Π - Circuit

Recall maths → $\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\Delta = a_{11}a_{22} - a_{12}a_{21}$

$$\boldsymbol{M}^{-1} = \frac{1}{\Delta} adj\boldsymbol{M} = \begin{bmatrix} {a_{22}}/{\Delta} & -\left({a_{12}}/{\Delta}\right) \\ -\left({a_{21}}/{\Delta}\right) & {a_{11}}/{\Delta} \end{bmatrix} \qquad adj\boldsymbol{M} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$



$$\begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{bmatrix} = \begin{bmatrix} \frac{L_2}{j\omega(L_1L_2 - M^2)} & \frac{-M}{j\omega(L_1L_2 - M^2)} \\ \frac{-M}{j\omega(L_1L_2 - M^2)} & \frac{L_1}{j\omega(L_1L_2 - M^2)} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix}$$



$$\text{mesh analysis} \rightarrow \begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{j\omega L_A} + \frac{1}{j\omega L_C} & -\frac{1}{j\omega L_C} \\ -\frac{1}{j\omega L_C} & \frac{1}{j\omega L_B} + \frac{1}{j\omega L_C} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix}$$

$$L_A = \frac{L_1L_2 - M^2}{L_2 - M}, \qquad L_B = \frac{L_1L_2 - M^2}{L_1 - M}, \qquad L_C = \frac{L_1L_2 - M^2}{M}$$

- Coupled Inductances
- Linear Transformers
- **Ideal Transformers**
- Three-Phase Transformers
- Transformer Applications

Ideal Transformers

Let's see… $k = 1 \leftarrow \mu \approx \infty$

$$(\mathbf{1}):\ \boldsymbol{V}_1 = j\omega L_1 \boldsymbol{I}_1 + j\omega M \boldsymbol{I}_2 \ \rightarrow \boldsymbol{I}_1 = \frac{(\boldsymbol{V}_1 - j\omega M \boldsymbol{I}_2)}{j\omega L_1}$$

$$(\mathbf{2}):\ \boldsymbol{V}_2 = j\omega M \boldsymbol{I}_1 + j\omega L_2 \boldsymbol{I}_2$$

$$(\mathbf{1} \rightarrow \mathbf{2}):\ \boldsymbol{V}_2 = j\omega L_2 \boldsymbol{I}_2 + \frac{M\boldsymbol{V}_1}{L_1} - \frac{j\omega M^2 \boldsymbol{I}_2}{L_1}$$

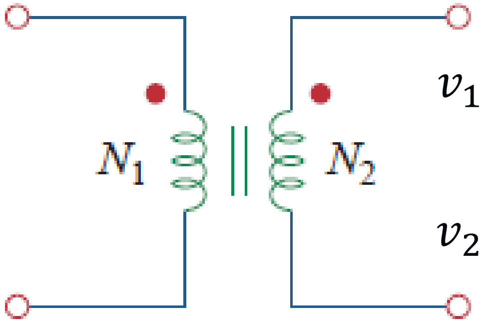
$$k = 1 \rightarrow M = \sqrt{L_1 L_2}$$

$$\boldsymbol{V}_2 = j\omega L_2 \boldsymbol{I}_2 + \frac{\sqrt{L_1 L_2}\boldsymbol{V}_1}{L_1} - \frac{j\omega L_1 L_2 \boldsymbol{I}_2}{L_1} = \sqrt{\frac{L_2}{L_1}}\boldsymbol{V}_1 = n\,\boldsymbol{V}_1$$

$$n\ (turns\ ratio) = constant, even\ if\, L_1\,, L_2\,, M \rightarrow \infty$$

Ideal transformer

- ❑ Coils have very large reactances $L_1\,, L_2\,, M \rightarrow \infty$
- ❑ Coupling coefficient is equal to unity $k = 1$
- ❑ Primary and secondary coils are lossless $R_1 = R_2 = 0$



$$v_1 = N_1 \frac{d\Phi}{dt} \qquad v_2 = N_2 \frac{d\Phi}{dt}$$

$$\frac{v_2}{v_1} = \frac{\boldsymbol{V}_2}{\boldsymbol{V}_1} = \frac{N_2}{N_1} = n$$

- ❑ Isolation transformer → n = 1, ($V_2 = V_1$)
- ❑ Step-up transformer → n > 1, ($V_2 > V_1$)
- ❑ Step-down transformer → n < 1, ($V_2 < V_1$)

Ideal Transformers

- Power conservation → $v_1 i_1 = v_2 i_2 \rightarrow \boldsymbol{V}_1 \boldsymbol{I}_1 = \boldsymbol{V}_2 \boldsymbol{I}_2$

$$\frac{\boldsymbol{I}_1}{\boldsymbol{I}_2} = \frac{\boldsymbol{V}_2}{\boldsymbol{V}_1} = n \rightarrow \frac{\boldsymbol{V}_2}{\boldsymbol{V}_1} = \frac{N_2}{N_1}, \qquad \frac{\boldsymbol{I}_2}{\boldsymbol{I}_1} = \frac{N_1}{N_2}$$

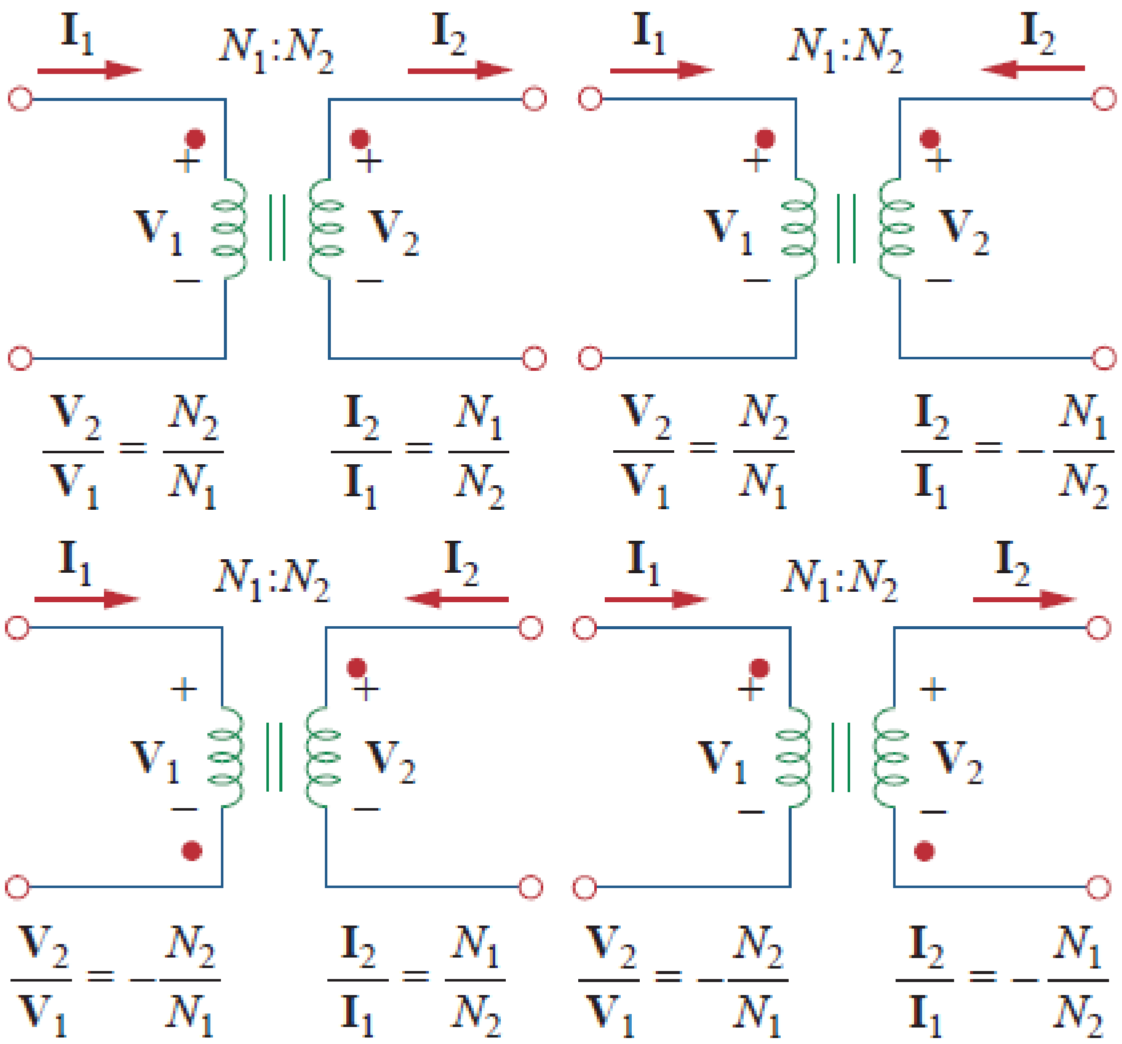
- Sign convention → *(four figures…)*

- Complex power …

$$\boldsymbol{S}_1 = \boldsymbol{V}_1 \boldsymbol{I}_1^* = \frac{\boldsymbol{V}_2}{n} (n\boldsymbol{I}_2)^* = \boldsymbol{V}_2 \boldsymbol{I}_2^* = \boldsymbol{S}_2$$

- Impedance transform
  (imp. matching i.e. for max PWR transfer)

$$\boldsymbol{Z}_{in} = \frac{\boldsymbol{V}_1}{\boldsymbol{I}_1} = \frac{1}{n^2} \frac{\boldsymbol{V}_2}{\boldsymbol{I}_2} = \frac{\boldsymbol{Z}_L}{n^2}$$



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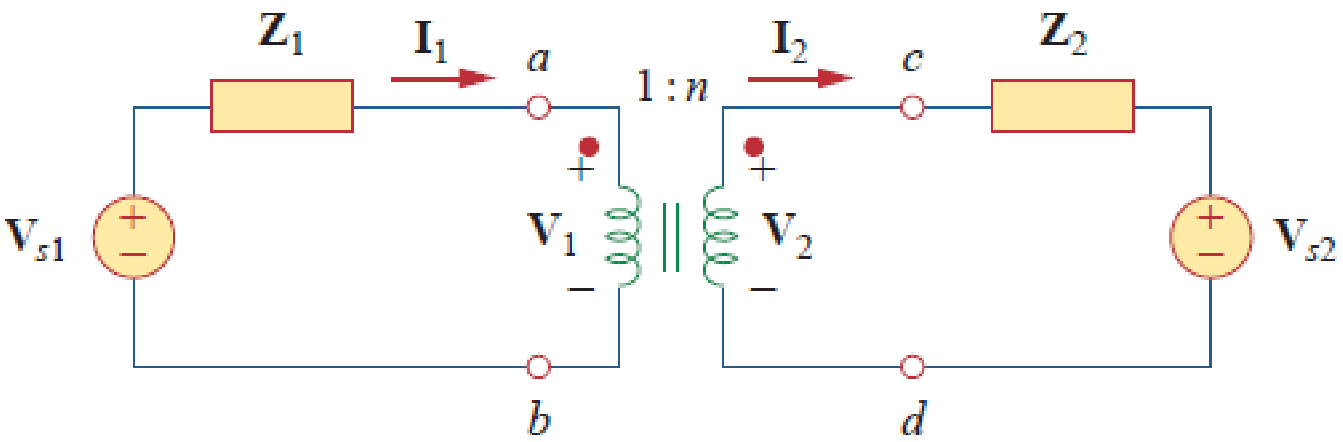
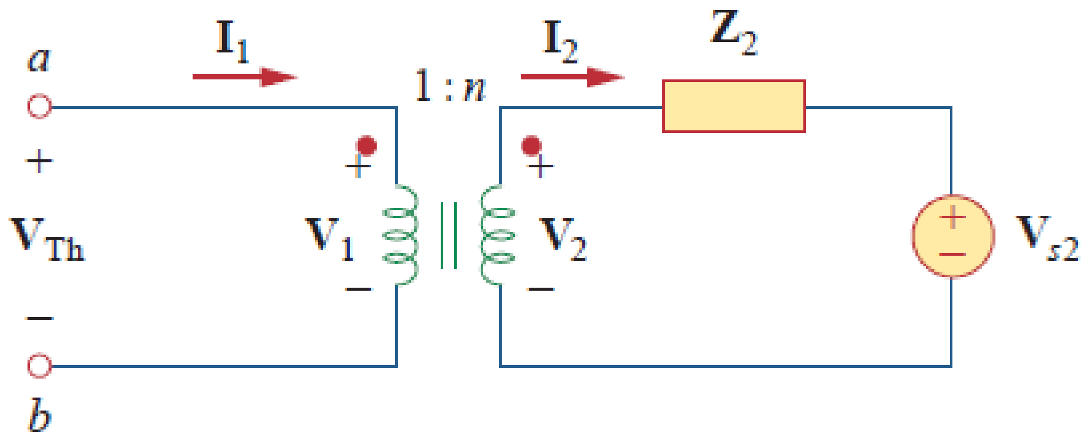
Ideal Transformers – Equivalent Circuit 1

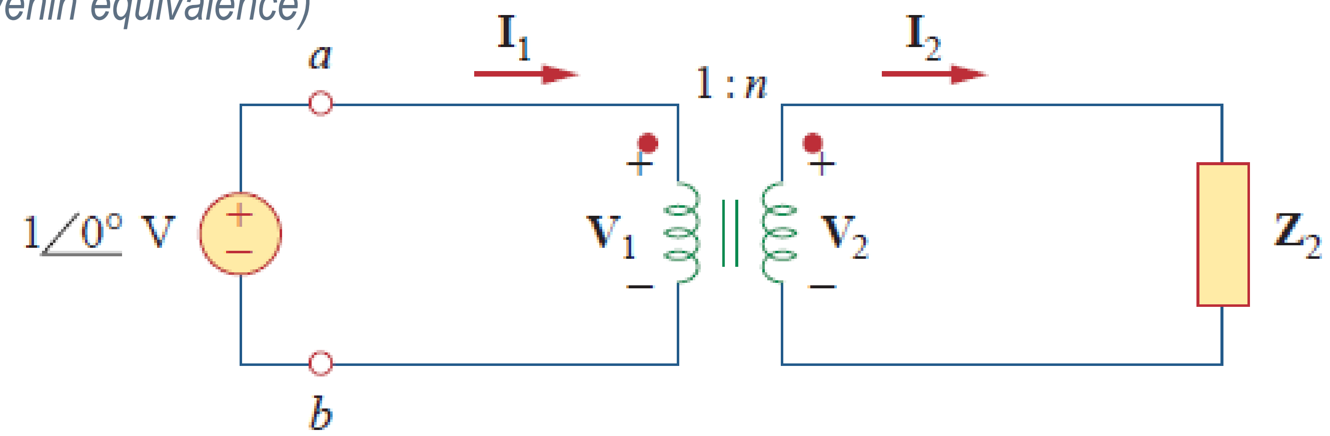




(1) Reflecting secondary to the primer circuit *(Thevenin equivalence)*

$$I_1 = 0 \rightarrow I_2 = 0 \rightarrow V_{Th} = V_1 = \frac{V_2}{n} = \frac{V_{S2}}{n}$$

$$Z_{Th} = \frac{V_1}{I_1} = \frac{V_2/n}{nI_2} = \frac{Z_2}{n^2}\,, \qquad V_2 = Z_2 I_2$$



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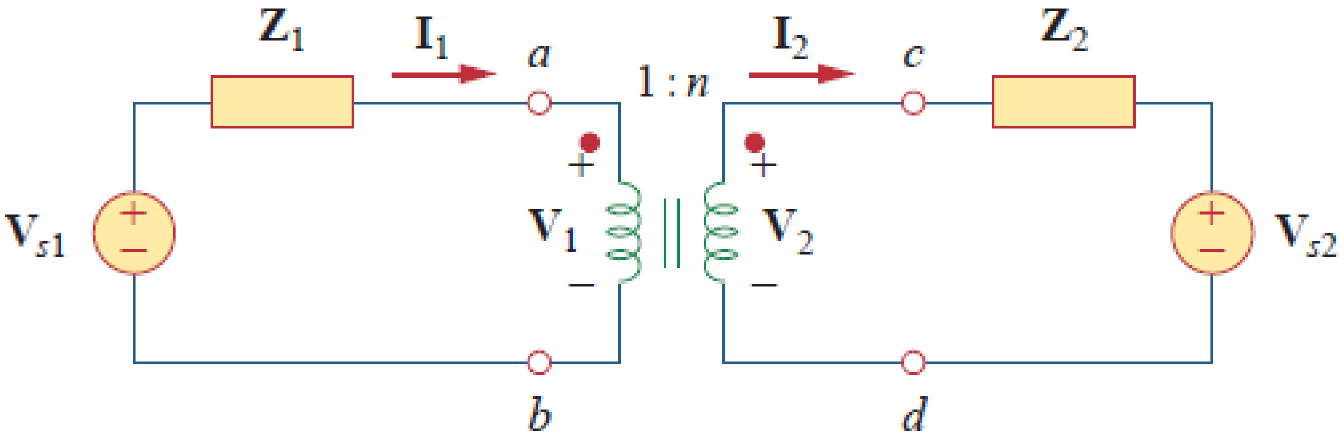
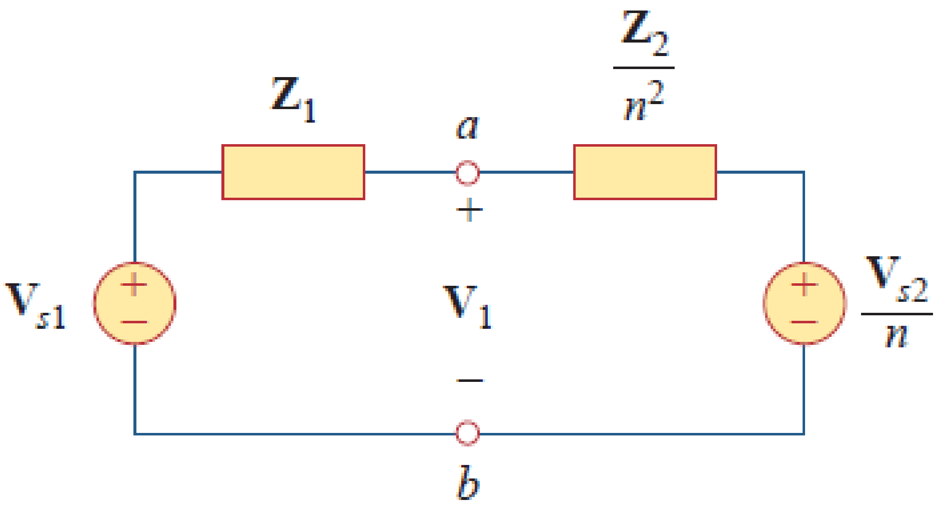
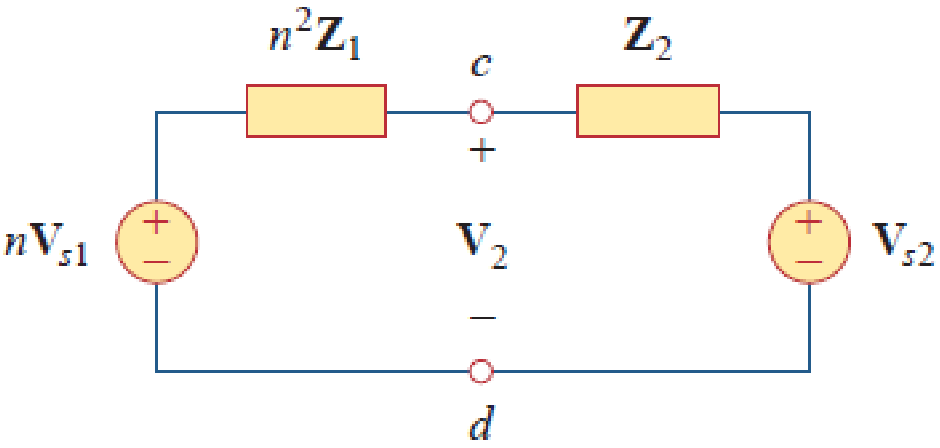
Ideal Transformers – Equivalent Circuit 2



(1) Reflecting secondary to the primer circuit (result)



(2) Reflecting primer to the secondary circuit



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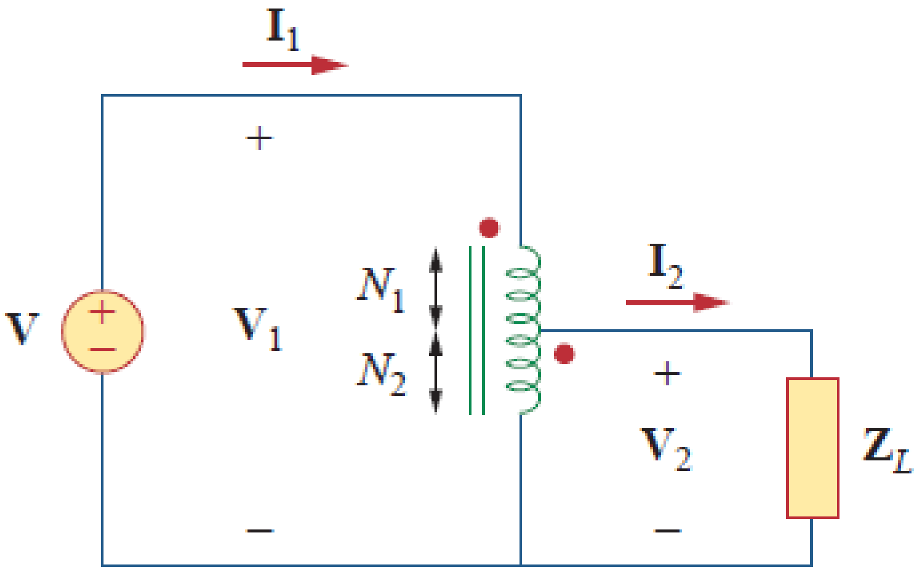
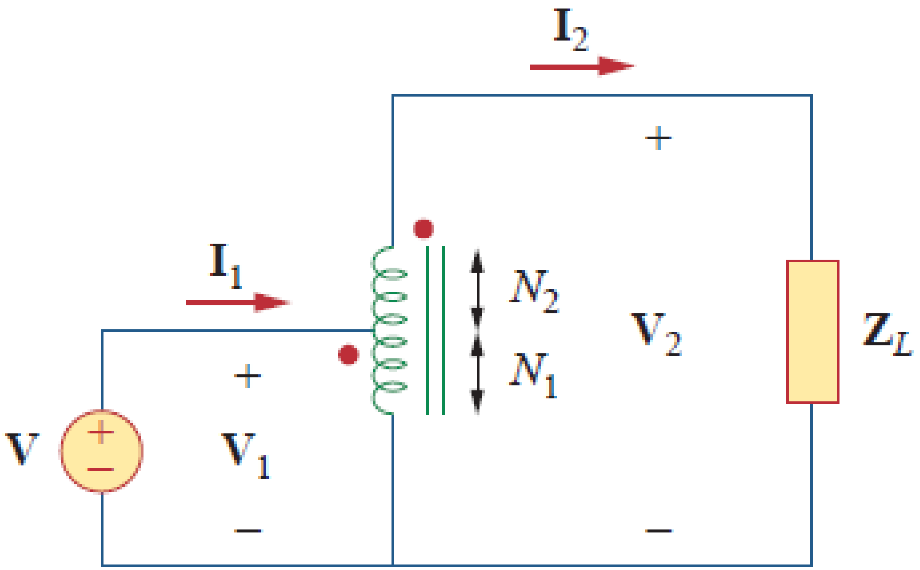
Ideal Autotransformers

Autotransformer → single coil with (variable) tap

- ❑ Electric isolation is lost
- ❑ Step-down transformer →← Step-up transformer





$$\frac{\boldsymbol{V}_1}{\boldsymbol{V}_2}=\frac{N_1+N_2}{N_2}=1+\frac{N_1}{N_2}\,,\qquad \boldsymbol{S}_1=\boldsymbol{S}_2\rightarrow\frac{\boldsymbol{I}_1}{\boldsymbol{I}_2}=\frac{N_2}{N_1+N_2}$$

$$\frac{\boldsymbol{V}_1}{\boldsymbol{V}_2}=\frac{N_1}{N_1+N_2}\,,\qquad \boldsymbol{S}_1=\boldsymbol{S}_2\rightarrow\frac{\boldsymbol{I}_1}{\boldsymbol{I}_2}=\frac{N_1+N_2}{N_1}=1+\frac{N_2}{N_1}$$

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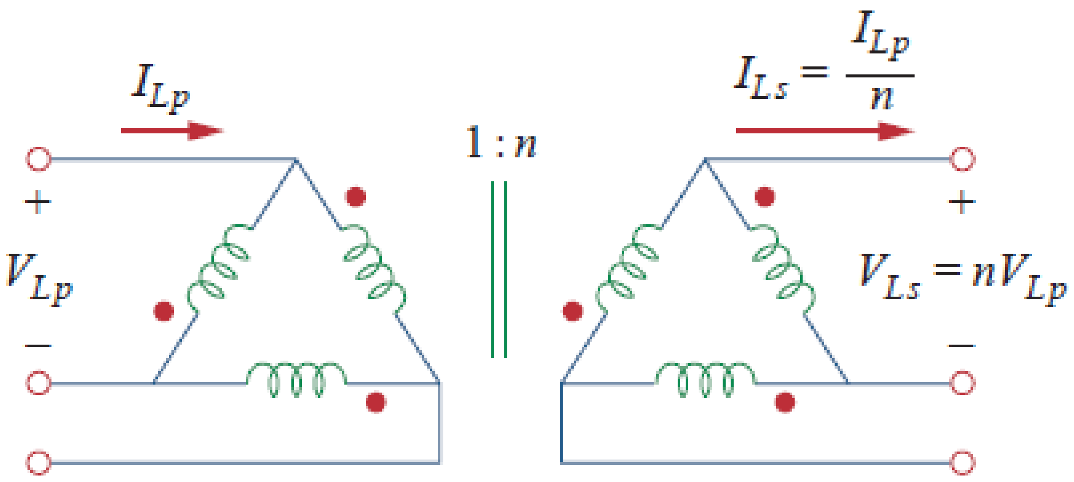
- ❑ **Three-Phase Transformers**
- ❑ Transformer Applications

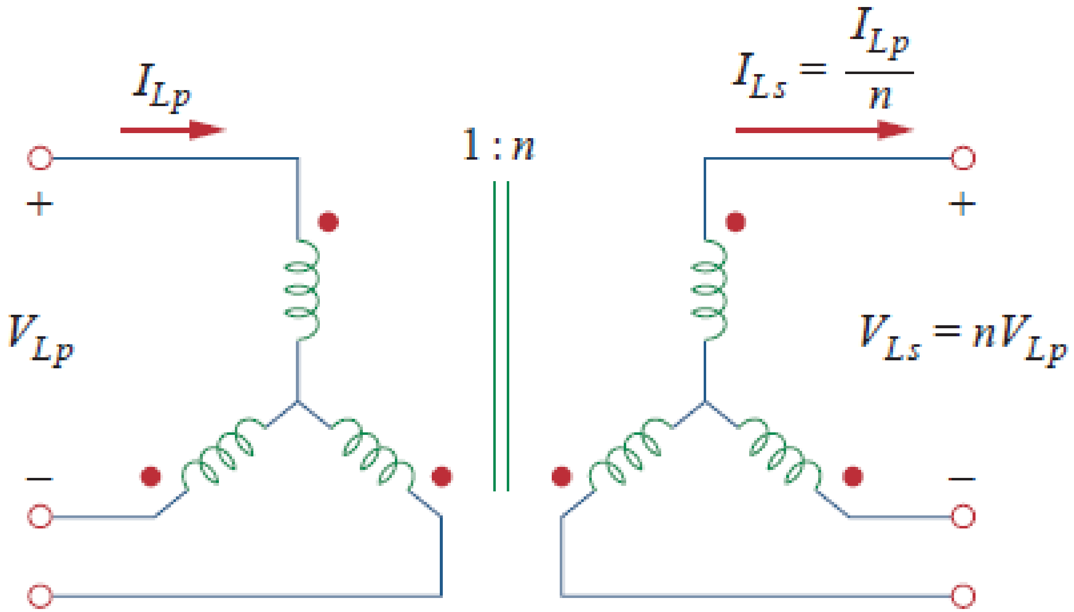
Three-Phase Transformers 1

Three-phase transformer

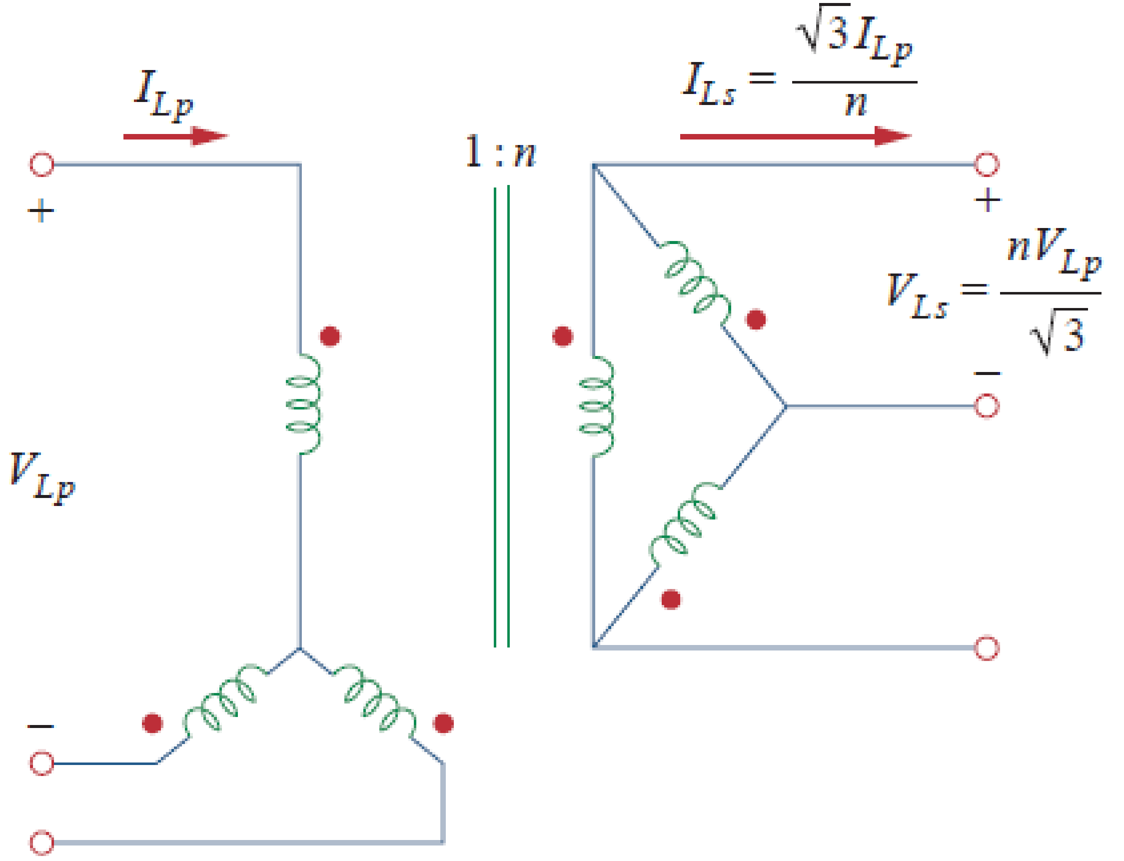
- ❑ <u>Smaller</u> and <u>cheaper</u> than *transformer bank*
- ❑ Four connections (*YY, DD, DY, YD*)
- ❑ For any connection $P_{total} = S_{total}\cos\Theta = \sqrt{3}V_L I_L \cos\Theta$ $\quad Q_{total} = S_{total}\sin\Theta = \sqrt{3}V_L I_L \sin\Theta$

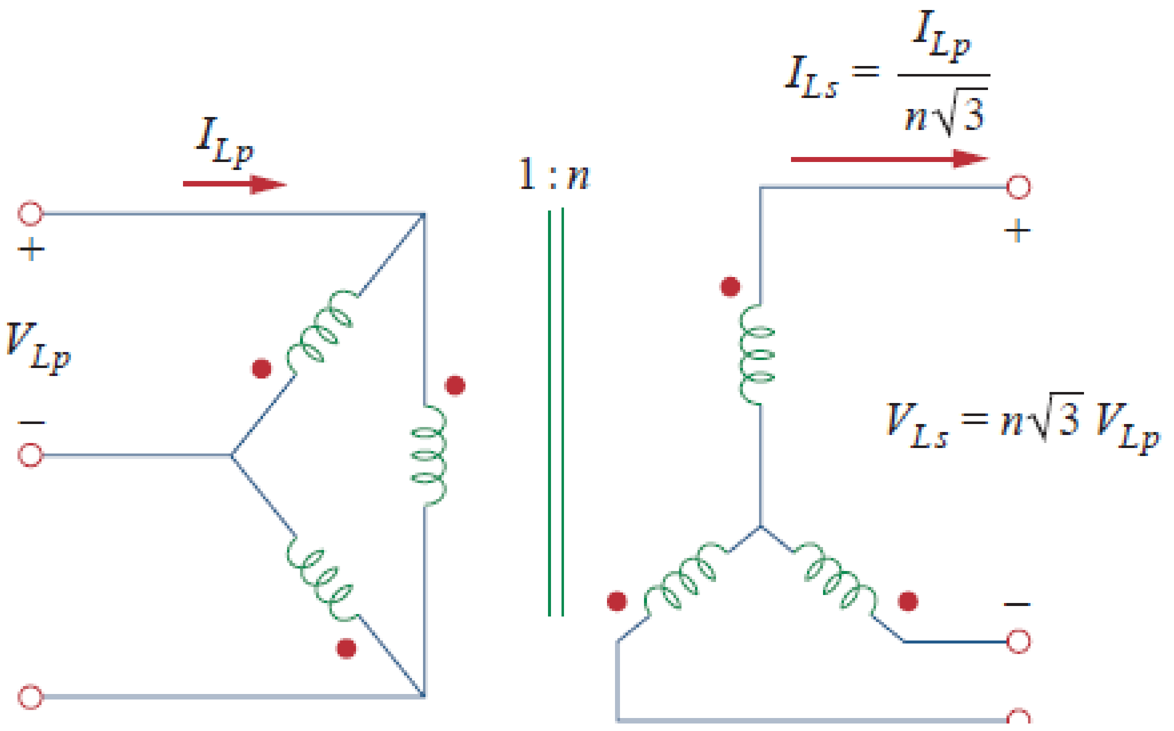
$$S_{total} = \sqrt{3}V_L I_L$$





Three-Phase Transformers 2





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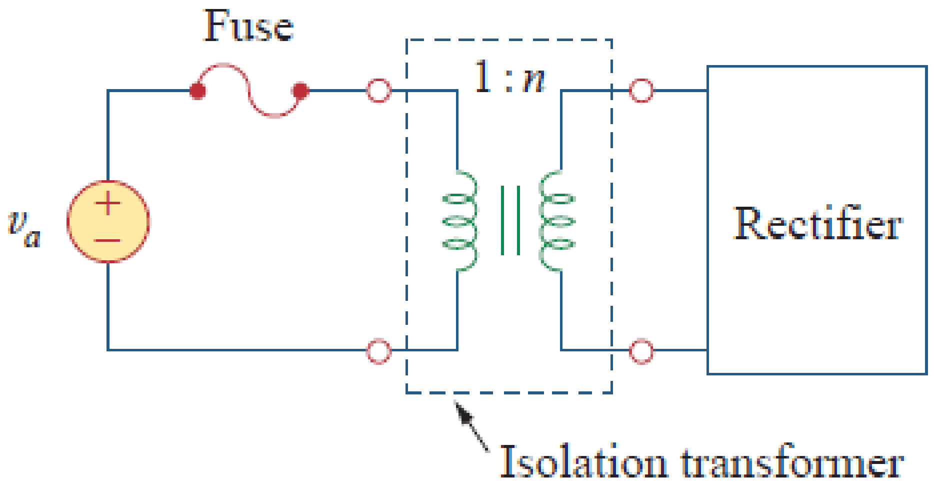
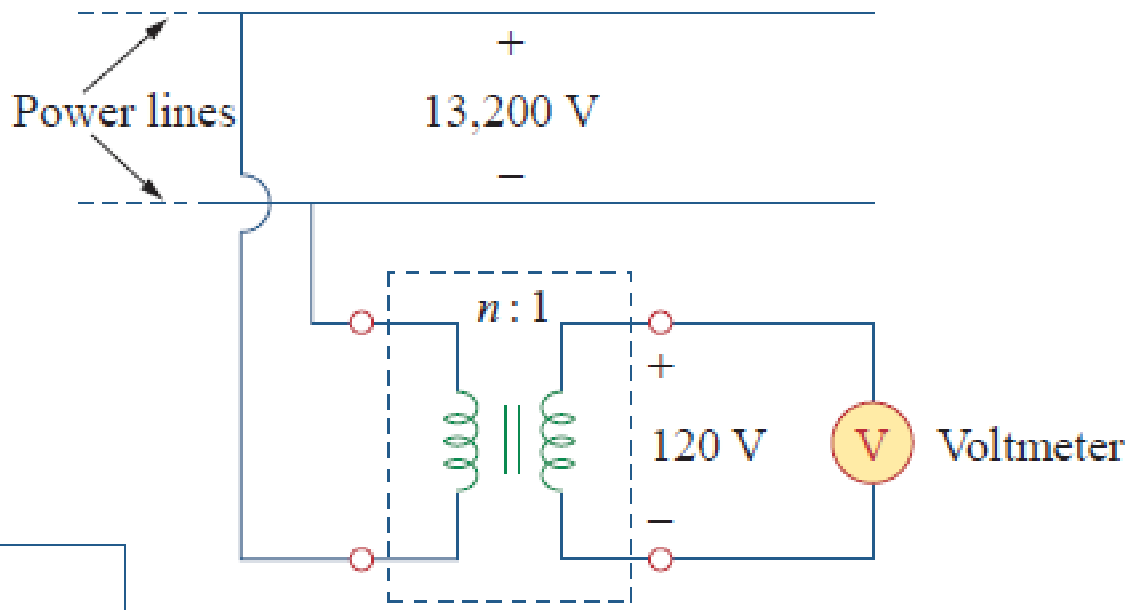
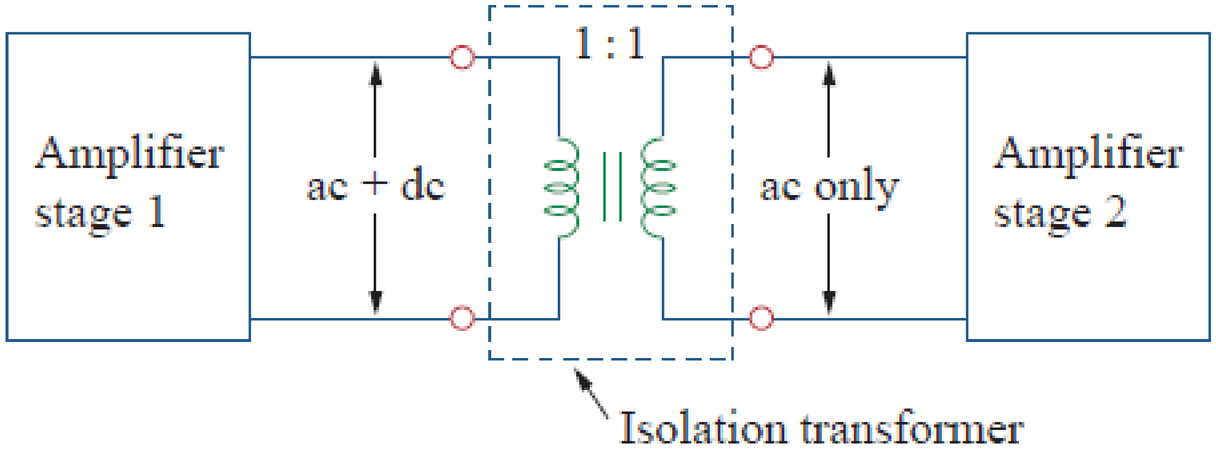
- ❑ **Transformer Applications**

App. – Isolation Device

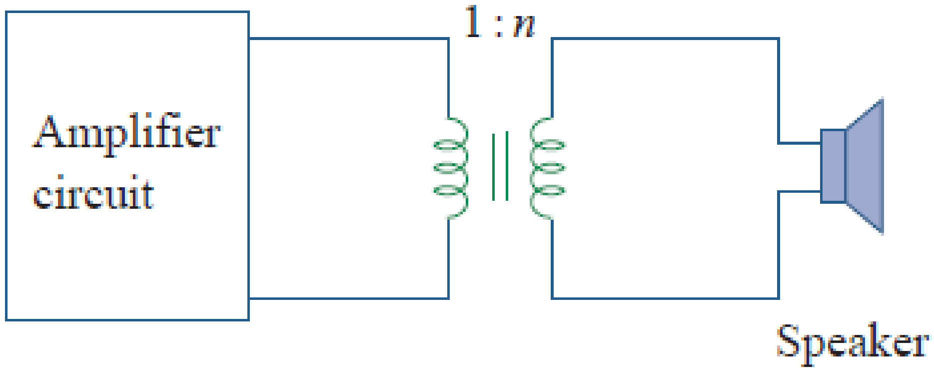




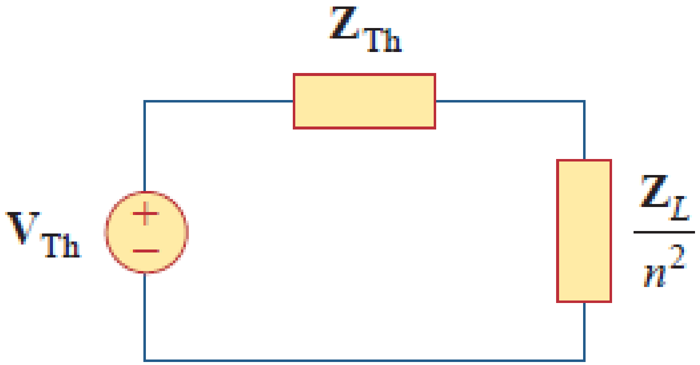


App. – Matching Device

Matching transformer for maximum power transfer



Equivalent circuit



$$\boldsymbol{Z}_{Th} = \frac{\boldsymbol{Z}_L}{n^2} \rightarrow n = \sqrt{\frac{\boldsymbol{Z}_L}{\boldsymbol{Z}_{Th}}}$$

$$I_p = \frac{\boldsymbol{V}_{Th}}{\boldsymbol{Z}_{Th} + \boldsymbol{Z}_L/n^2} \rightarrow I_s = \frac{I_p}{n} = \frac{\boldsymbol{V}_{Th}/n}{\boldsymbol{Z}_{Th} + \boldsymbol{Z}_L/n^2}$$

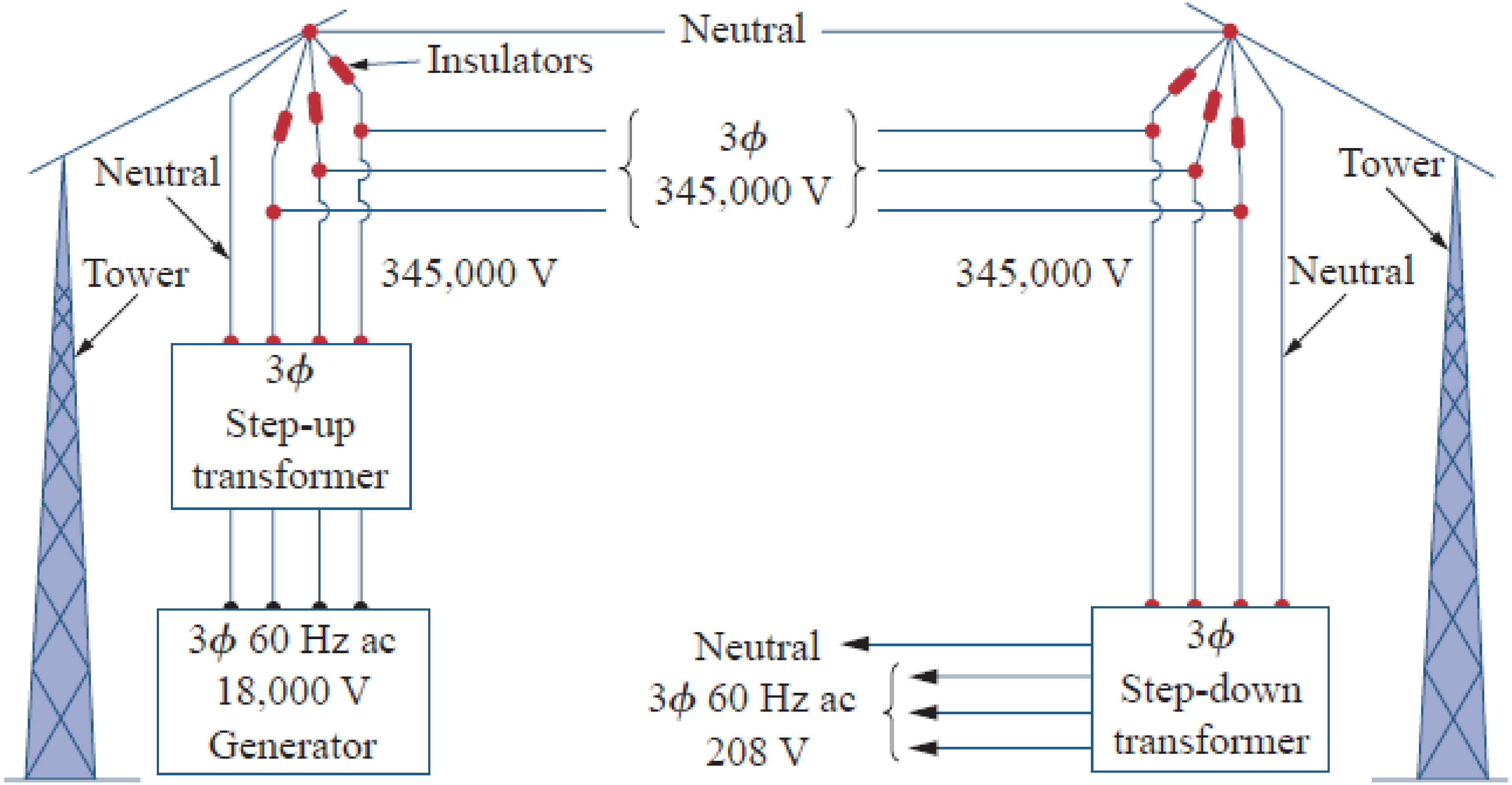
- ❑ With matching transformer…

$$P_L = I_s^2 \boldsymbol{Z}_L = \left(\frac{\boldsymbol{V}_{Th}/n}{\boldsymbol{Z}_{Th} + \boldsymbol{Z}_L/n^2}\right)^2 \boldsymbol{Z}_L = \left(\frac{n\boldsymbol{V}_{Th}}{n^2\boldsymbol{Z}_{Th} + \boldsymbol{Z}_L}\right)^2 \boldsymbol{Z}_L$$

- ❑ Without matching transformer…

$$P_L = I^2 \boldsymbol{Z}_L = \left(\frac{\boldsymbol{V}_{Th}}{\boldsymbol{Z}_{Th} + \boldsymbol{Z}_L}\right)^2 \boldsymbol{Z}_L$$

App. - Power Distribution



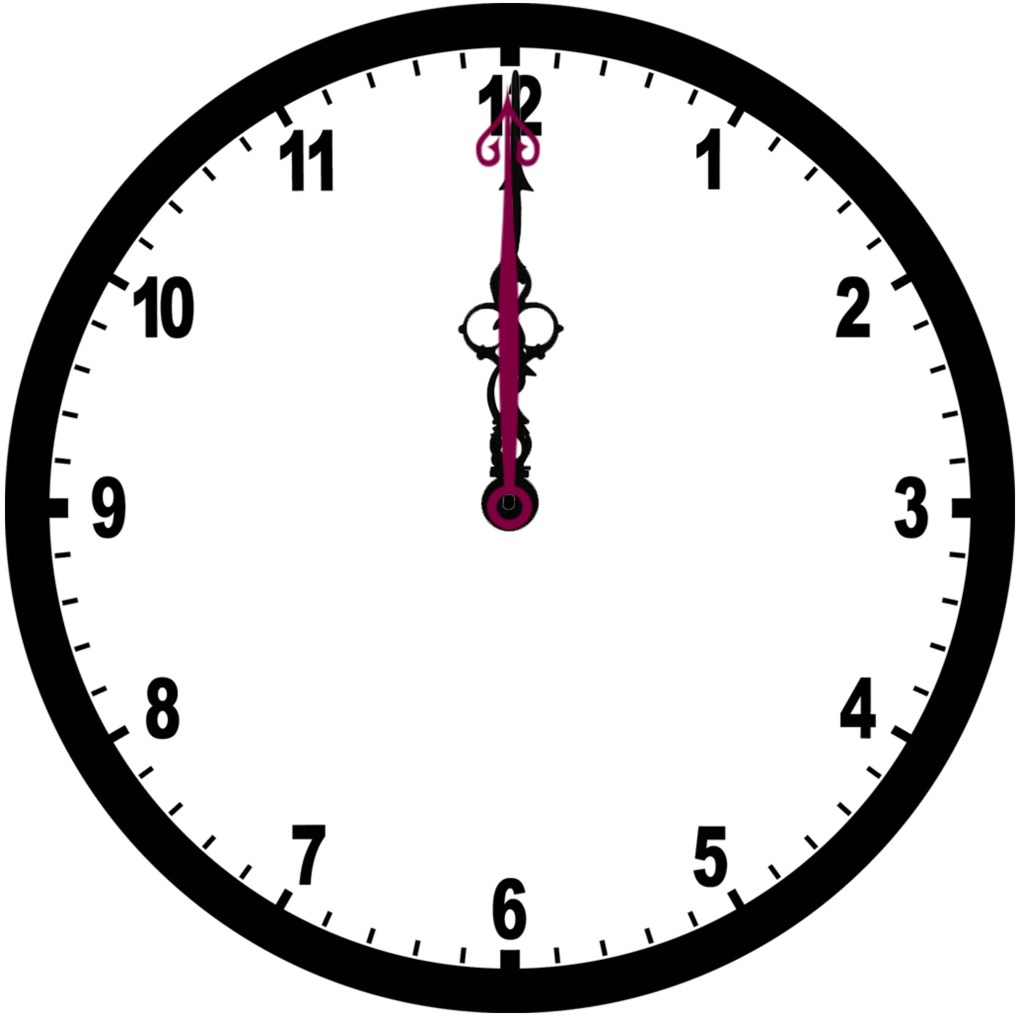
$$P_{loss} = I^2 R_w = \frac{(\Delta V)^2}{R_w}$$

$$\Delta V = V_{send} - V_{rec}$$

Step-up transforming...

$$V_{send} \approx V_{rec} \rightarrow P_{loss} \approx 0$$

Questions



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