

DR. GYURCSEK ISTVÁN

Multi-Wave Signals and Circuits

The Fourier Series

Sources and additional materials (recommended)

- ❑ *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- ❑ *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- ❑ *Simonyi K.: Villamosságtan. AK Budapest 1983, ISBN:9630534134*
- ❑ *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- ❑ *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*
- ❑ *Zombory L.: Elektromágneses terek. MK Budapest 2006, (www.electro.uni-miskolc.hu)*



- Trigonometric Form of Fourier Series**
- Amplitude-Phase Form of Fourier Series
- Exponential Form of Fourier Series
- Circuit Applications
- Average Power and RMS Values

Trigonometric Fourier Series



$$f(t) = f(t + nT_0) \leftarrow n = 1, 2, 3, \dots$$

Fourier's theorem – Under suitable conditions any periodic function can be represented by a Fourier series.

$$f(t) = A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t) \leftarrow \omega_0 = \frac{2\pi}{T_0}$$

Fourier series: $f(t) \rightarrow$ DC component + AC component (infinite series of sinusoids).

Convergent (requirements \rightarrow Dirichlet conditions; sufficient, not necessary)

- $f(t)$ is single-valued function everywhere.
- $f(t)$ has a finite number of finite discontinuities in one period.
- $f(t)$ has a finite number of maxima and minima in one period.
- The integral $\int_{t_0}^{t_0+T} |f(t)| dt < \infty$ for any t_0

Note – Theorem published by Fourier (1822), acceptable proof later by Dirichlet.

Fourier Analysis



- ❑ Discrete Fourier analysis → process, determining the coefficients in Fourier series
- ❑ Continuous Fourier analysis → process, determining Fourier transform

Before we start ... $\int_0^T \sin n\omega t \cos m\omega t dt = 0$

$$\int_0^T \sin n\omega t dt = 0 \quad \int_0^T \sin n\omega t \sin m\omega t dt = 0 \quad (n \neq m), \quad \text{and} \quad \int_0^T \sin^2 n\omega t dt = \frac{T}{2}$$
$$\int_0^T \cos n\omega t dt = 0 \quad \int_0^T \cos n\omega t \cos m\omega t dt = 0 \quad (n \neq m), \quad \text{and} \quad \int_0^T \cos^2 n\omega t dt = \frac{T}{2}$$

Fourier Analysis $(A_0 =?, A_k =?, B_k =?)$



$$\int_0^T f(t) dt = \int_0^T A_0 dt + \sum_{k=1}^{\infty} \left(A_k \int_0^T \cos k\omega_0 t dt + B_k \int_0^T \sin k\omega_0 t dt \right) \quad (A_0 =?, \quad A_k =?, \quad B_k =?)$$

$$\int_0^T f(t) dt = \int_0^T A_0 dt = A_0 T \quad A_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$\int_0^T f(t) \cos m\omega_0 t dt = \int_0^T A_0 \cos m\omega_0 t dt + \sum_{k=1}^{\infty} \left(A_k \int_0^T \cos k\omega_0 t \cos m\omega_0 t dt + B_k \int_0^T \sin k\omega_0 t \cos m\omega_0 t dt \right)$$

$$\int_0^T f(t) \cos m\omega_0 t dt = A_k \frac{T}{2}, \quad (k = m) \quad A_k = \frac{2}{T} \int_0^T f(t) \cos k\omega_0 t dt$$

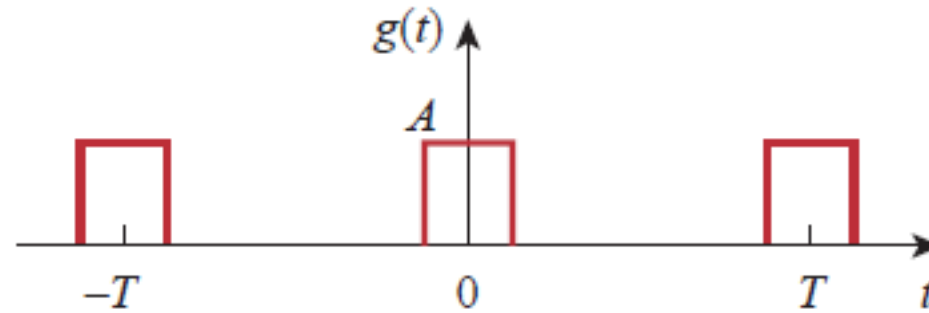
Homework \rightarrow prove that

$$B_k = \frac{2}{T} \int_0^T f(t) \sin k\omega_0 t dt$$

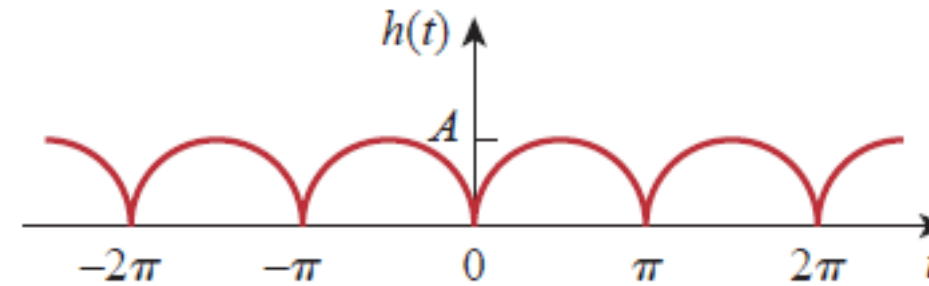
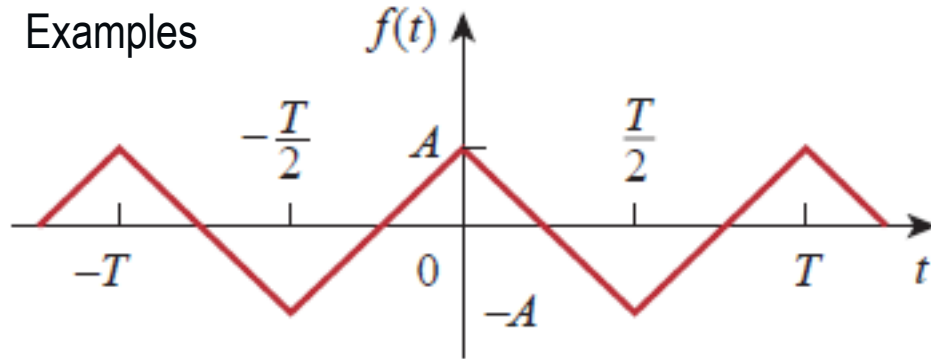
Symmetry Considerations



Even symmetry $f(t) = (f(-t))$



Examples

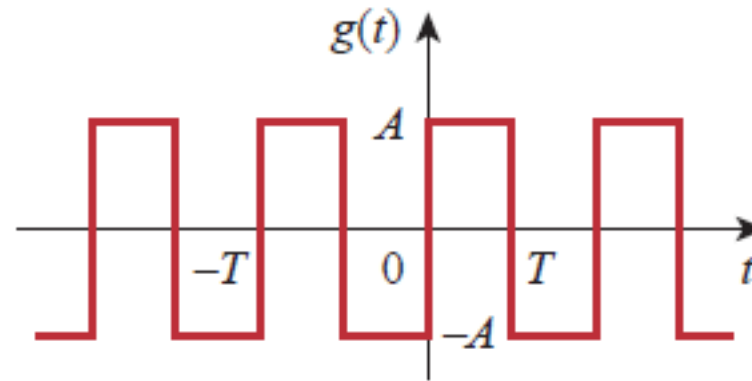


Main property $\rightarrow \int_{-T/2}^{T/2} f(t) dt = 2 \int_0^{T/2} f(t) dt \rightarrow A_0 = \frac{2}{T} \int_0^{T/2} f(t) dt \quad A_k = \frac{4}{T} \int_0^{T/2} f(t) \cos k\omega_0 t dt \quad B_k = 0$

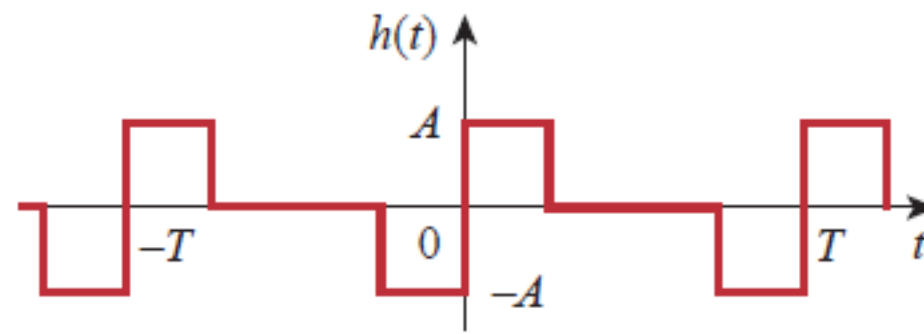
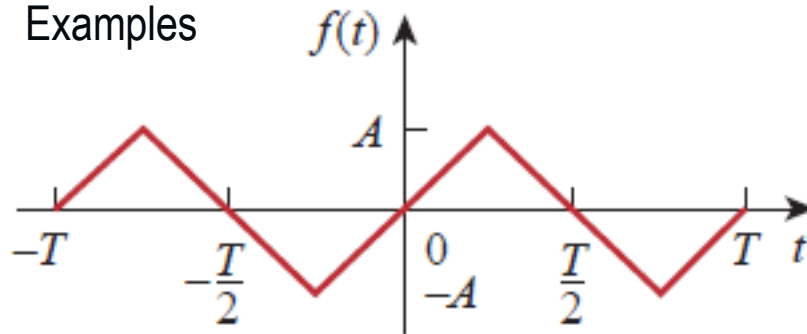
Symmetry Considerations



Odd symmetry $f(-t) = -f(t)$

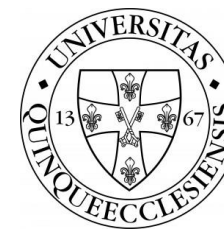


Examples

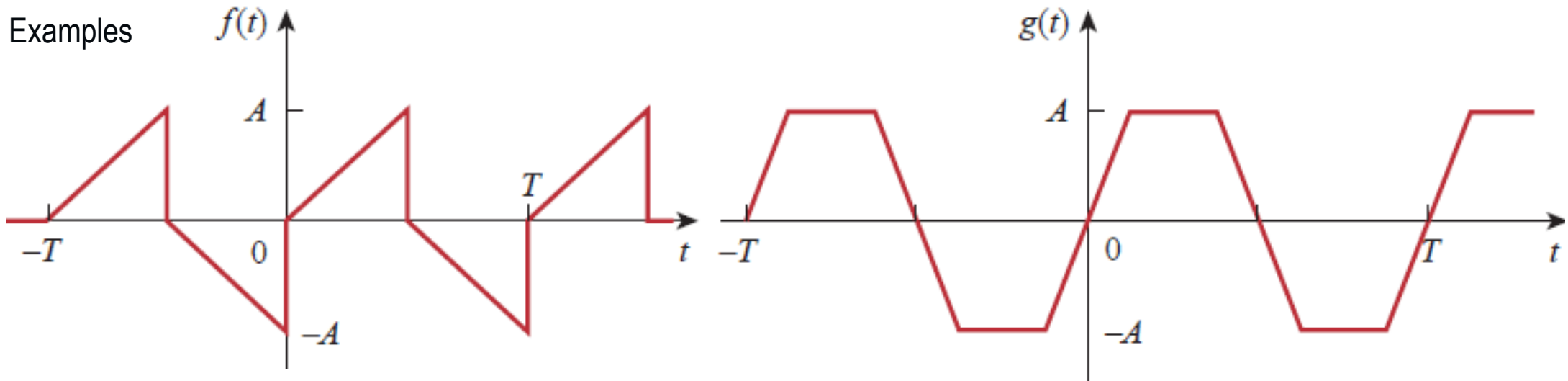


Main property $\rightarrow \int_{-T/2}^{T/2} f(t) dt = 0 \rightarrow A_0 = 0 \quad A_k = 0 \quad B_k = \frac{4}{T} \int_0^{T/2} f(t) \sin k\omega_0 t dt$

Symmetry Considerations



Half-wave (odd) symmetry $f(t - T/2) = -(f(t))$... next half-wave is the 'mirror'

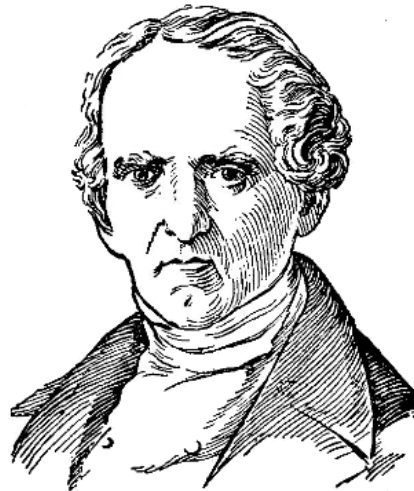


$$A_0 = 0$$

$$A_k = \begin{cases} \frac{4}{T} \int_0^{T/2} f(t) \cos k\omega_0 t dt & k = \text{odd} \\ 0 & k = \text{even} \end{cases}$$

$$B_k = \begin{cases} \frac{4}{T} \int_0^{T/2} f(t) \sin k\omega_0 t dt & k = \text{odd} \\ 0 & k = \text{even} \end{cases}$$

Fourier Analysis (Summary)



Fourier

$$f(t) = f(t + nT)$$

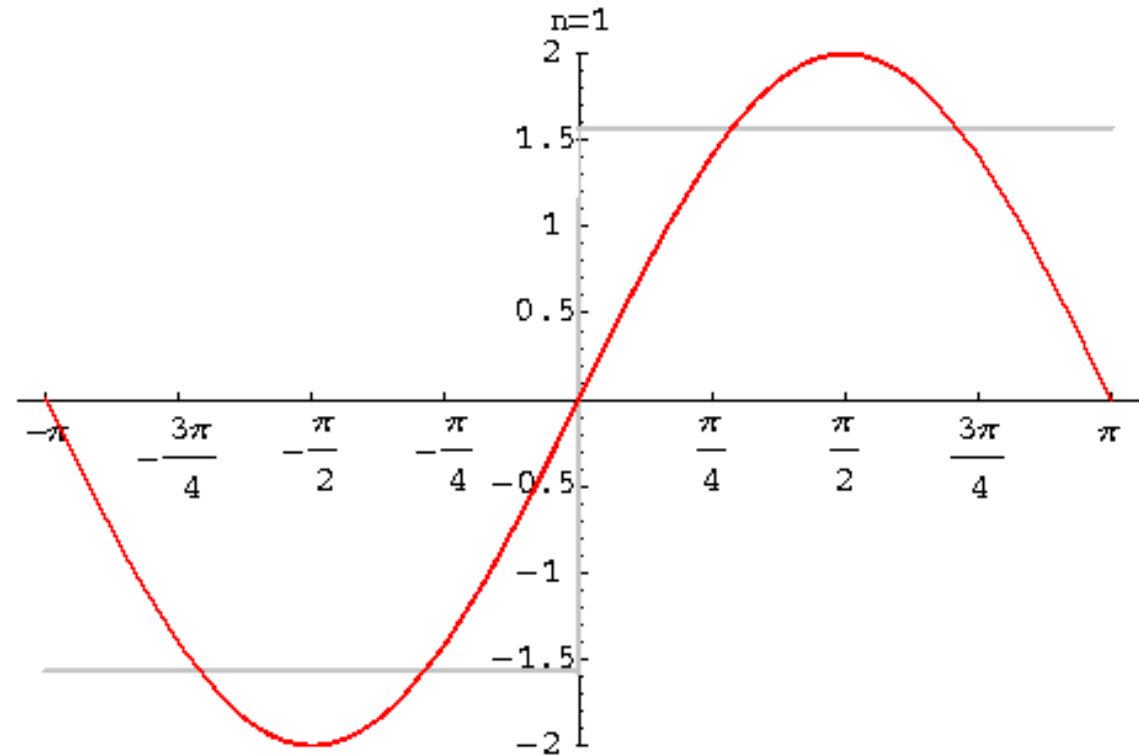
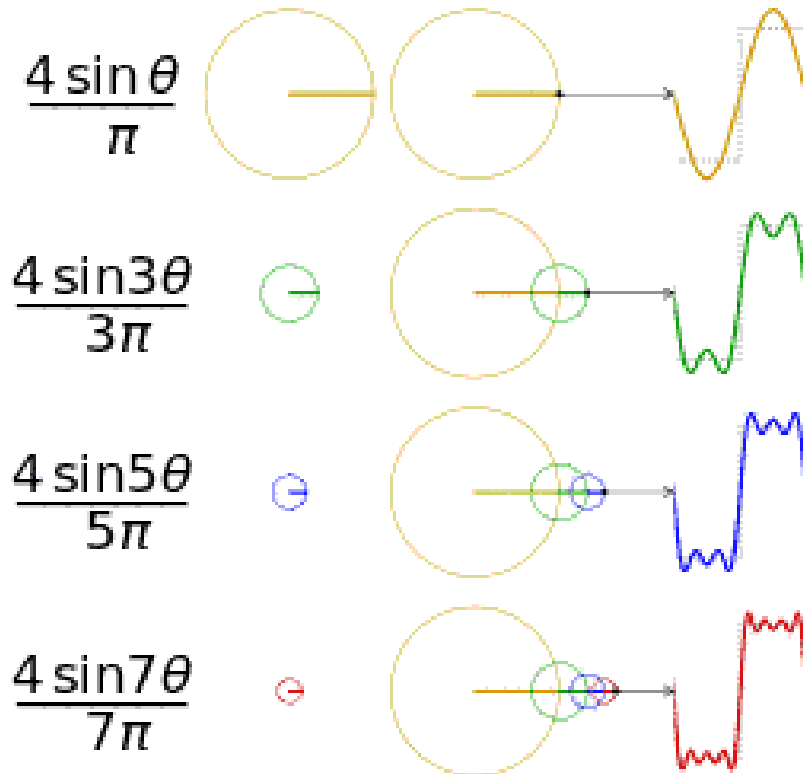
$$f(t) = A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$$

$$A_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$A_k = \frac{2}{T} \int_0^T f(t) \cos k\omega_0 t dt$$

$$B_k = \frac{2}{T} \int_0^T f(t) \sin k\omega_0 t dt$$

Illustration





- Trigonometric Form of Fourier Series
- Amplitude-Phase Form of Fourier Series**
- Exponential Form of Fourier Series
- Circuit Applications
- Average Power and RMS Values

Amplitude-Phase Form



$$f(t) = A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$$

$$f(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t + \varphi_k)$$

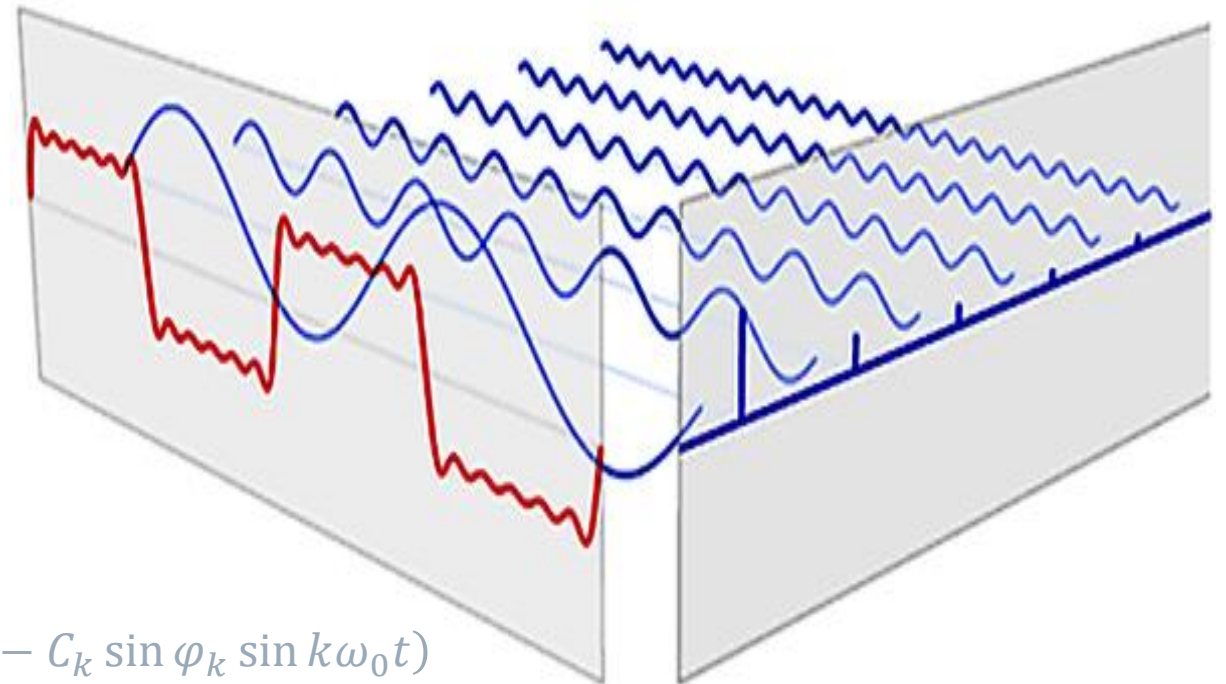
$$C_0 = A_0$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sum_{k=1}^{\infty} C_k \cos(k\omega_0 t + \varphi_k) = \sum_{k=1}^{\infty} (C_k \cos \varphi_k \cos k\omega_0 t - C_k \sin \varphi_k \sin k\omega_0 t)$$

$$A_k = C_k \cos \varphi_k, \quad B_k = -C_k \sin \varphi_k$$

$$C_k = \sqrt{A_k^2 + B_k^2}, \quad \varphi_k = -\tan^{-1} \frac{B_k}{A_k}$$



Frequency spectrum → plots

- amplitudes of the harmonics versus frequency
- phases of the harmonics versus frequency.



- ❑ Trigonometric Form of Fourier Series
- ❑ Amplitude-Phase Form of Fourier Series
- ❑ **Exponential Form of Fourier Series**
- ❑ Circuit Applications
- ❑ Average Power and RMS Values

Exponential Form



$$\left. \begin{aligned} \cos k\omega_0 t &= \frac{e^{jk\omega_0 t} + e^{-jk\omega_0 t}}{2} \\ \sin k\omega_0 t &= \frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{j2} \end{aligned} \right\} \rightarrow f(t) = A_0 + \frac{1}{2} \sum_{k=1}^{\infty} [(A_k - jB_k)e^{jk\omega_0 t} + (A_k + jB_k)e^{-jk\omega_0 t}]$$

$$D_0 = A_0, \quad \mathbf{D}_k = \frac{(A_k - jB_k)}{2}, \quad \mathbf{D}_{-k} = \mathbf{D}_k^* = \frac{(A_k + jB_k)}{2} \rightarrow f(t) = D_0 + \sum_{k=1}^{\infty} (\mathbf{D}_k e^{jk\omega_0 t} + \mathbf{D}_{-k} e^{-jk\omega_0 t})$$

$$f(t) = \sum_{k=-\infty}^{\infty} \mathbf{D}_k e^{jk\omega_0 t} \quad \rightarrow \mathbf{D}_k = \frac{1}{T} \int_0^T f(t) e^{-jk\omega_0 t} dt$$

Exponential Fourier series of a periodic $f(t)$ describes the spectrum of $f(t)$
(amplitude and phase of ac components *at positive and negative harmonic frequencies*).

Relationship between Three Forms



□ Trigonometric Form of Fourier Series $f(t) = A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$

□ Amplitude-Phase Form of Fourier Series $f(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t + \varphi_k)$

□ Exponential Form of Fourier Series $f(t) = \sum_{k=-\infty}^{\infty} D_k e^{jk\omega_0 t}$

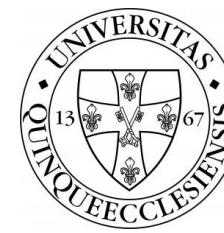
$$A_k = C_k \cos \varphi_k, \quad B_k = -C_k \sin \varphi_k, \quad C_k = \sqrt{A_k^2 + B_k^2}, \quad \varphi_k = -\tan^{-1} \frac{B_k}{A_k}$$

$$D_k = D_k e^{j\Theta_k} = \frac{(A_k - jB_k)}{2} = \frac{C_k}{2} e^{j\varphi_k} \rightarrow D_k = \frac{\sqrt{A_k^2 + B_k^2}}{2}, \quad \Theta_k = -\tan^{-1} \frac{B_k}{A_k} = \varphi_k$$



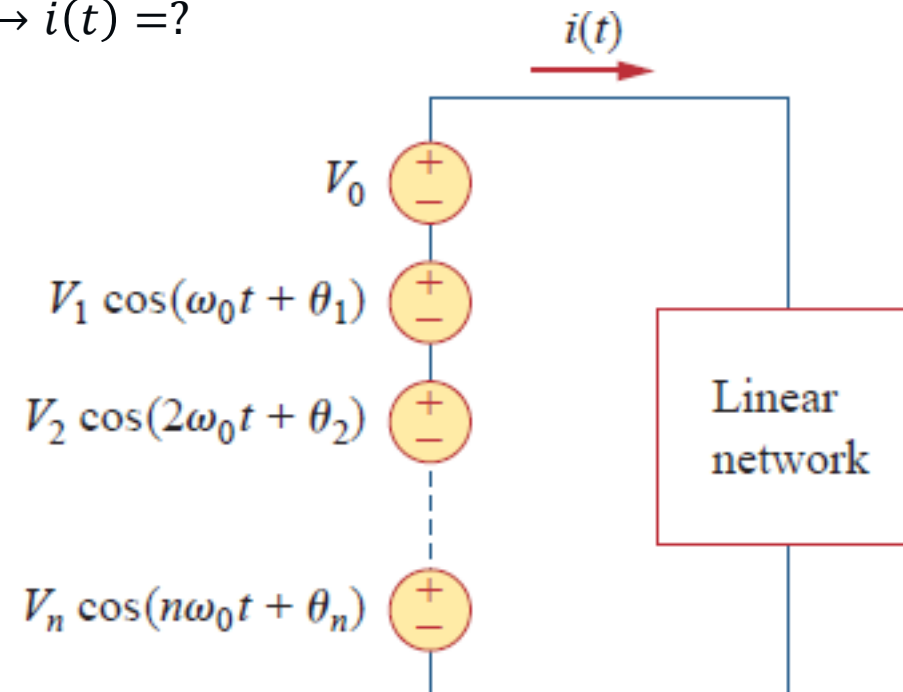
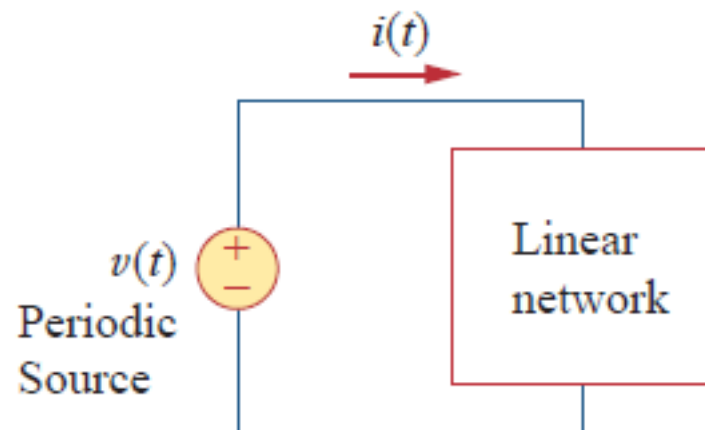
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Circuit Application Procedure

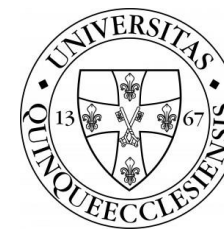


Excitation $v(t)$ is a periodic source. Task $\rightarrow i(t) = ?$

$$v(t) = V_0 + \sum_{k=1}^{\infty} V_k \cos(k\omega_0 t + \Theta_k)$$



Circuit Application Procedure

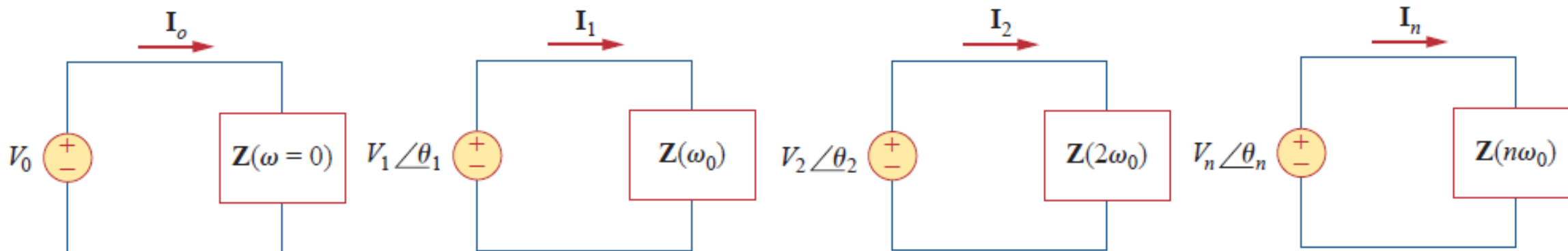


(STEP 1) Express the excitation as a Fourier series

$$v(t) = V_0 + \sum_{k=1}^{\infty} V_k \cos(k\omega_0 t + \Theta_k)$$

(STEP 2) Find the response of each term in the series.

$$v_k(t) \rightarrow V_k \rightarrow I_k = \frac{V_k}{Z_k} \rightarrow i_k(t), \quad k = 0, 1, 2, \dots$$



(STEP 3) Apply the superposition principle

$$i(t) = I_0 + i_1(t) + i_2(t) + \dots = I_0 + \sum_{k=1}^{\infty} I_k \cos(k\omega_0 t + \varphi_k)$$



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Average Power



$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \varphi_{Vn})$$

$$i(t) = I_0 + \sum_{m=1}^{\infty} I_m \cos(m\omega_0 t + \varphi_{Im})$$

$$P = \frac{1}{T} \int_0^T v(t)i(t)dt$$

$$P = \frac{1}{T} \int_0^T V_0 I_0 dt + \sum_{m=1}^{\infty} \frac{V_0 I_m}{T} \int_0^T \cos(m\omega_0 t + \varphi_{Im}) dt$$

← zero!

$$+ \sum_{n=1}^{\infty} \frac{V_n I_0}{T} \int_0^T \cos(n\omega_0 t + \varphi_{Un}) dt$$

← zero!

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{V_n I_m}{T} \int_0^T \cos(n\omega_0 t + \varphi_{Un}) \cos(m\omega_0 t + \varphi_{Im}) dt$$

← n = m → non-zero

$$P = V_0 I_0 + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\varphi_{Vn} - \varphi_{In})$$

$$P = \sum_{n=0}^{\infty} P_n$$

Total average power → Sum of the average powers in each harmonically related voltage and current.

RMS Values



$$F_{rms} = \sqrt{\frac{1}{T} \int_0^T f^2(t) dt}$$

$$F_{rms}^2 = \frac{1}{T} \int_0^T C_0^2 dt + 2 \sum_{n=1}^{\infty} \frac{C_0 C_n}{T} \int_0^T \cos(n\omega_0 t + \varphi_n) dt + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{C_n C_m}{T} \int_0^T \cos(n\omega_0 t + \varphi_n) \cos(m\omega_0 t + \varphi_m) dt$$

$$F_{rms} = \sqrt{C_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2}$$

$$F_{rms} = \sqrt{C_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (\sqrt{2} C_{rmsn})^2} = \sqrt{C_0^2 + \sum_{n=1}^{\infty} C_{rmsn}^2}$$

$$F_{rms} = \sqrt{A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (A_n^2 + B_n^2)}$$

$$F_{rms}^2 = \sum_{n=0}^{\infty} F_{rmsn}^2$$

Parseval's Theorem for Periodic Functions



$$F_{rms}^2 = A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (A_n^2 + B_n^2)$$

Marc-Antoine Parseval
French mathematician (1755 - 1836)



$$f(t) = i(t) \rightarrow P = F_{rms}^2 R, \quad f(t) = v(t) \rightarrow P = \frac{F_{rms}^2}{R}$$

To avoid nature of signal (current or voltage) $\rightarrow R = 1 \Omega \rightarrow P_{1 \Omega} = F_{rms}^2 = A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (A_n^2 + B_n^2)$

Parseval's theorem \rightarrow average power in a periodic signal is the sum of

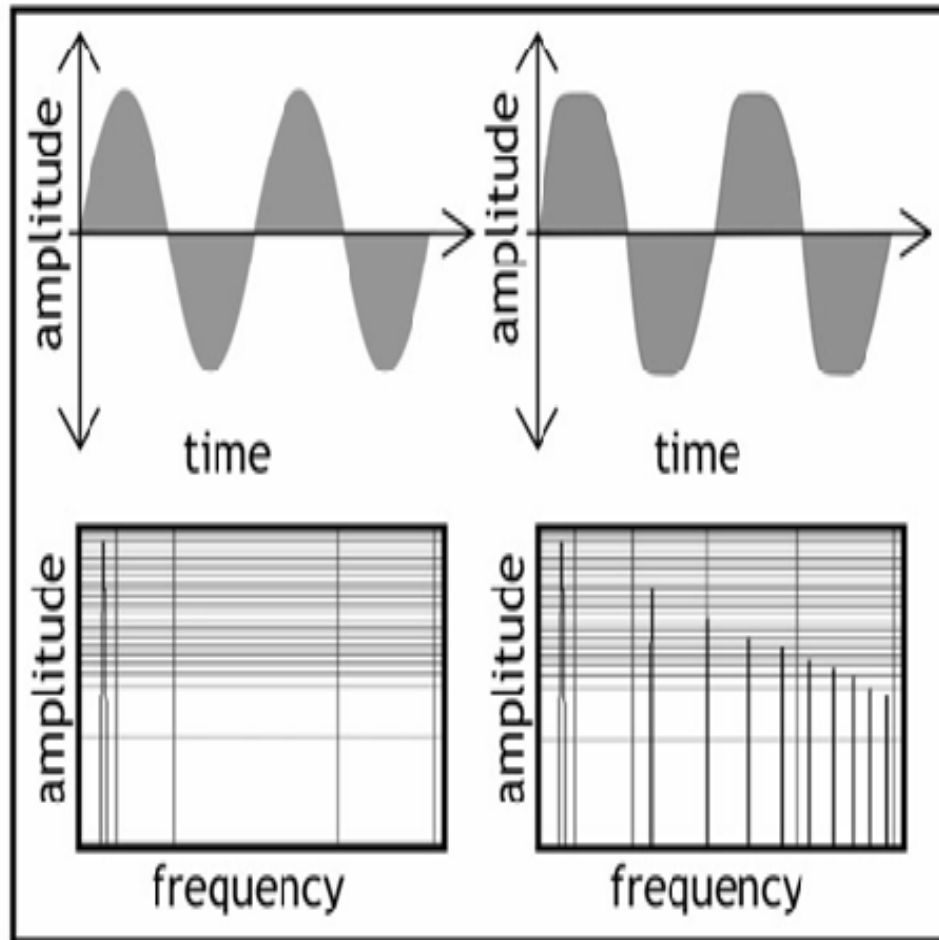
- average power in its dc component and
- average powers in its harmonics.

$$P = P_0 + \sum_{n=1}^{\infty} P_n$$

Parseval's theorem (in general)

- Energy of periodic signals \rightarrow concentrated at the harmonic components
- Energy of nonperiodic signals \rightarrow spread over the entire spectrum (Fourier transform ... later on)

Recall: Specific Factors



□ RMS Value

$$\rightarrow V_{rms} = V = \sqrt{\frac{1}{T} \int_0^T v^2(t) dt}$$

□ D Factor

$$\rightarrow D_{Vn} = \frac{V_n}{V_1}$$

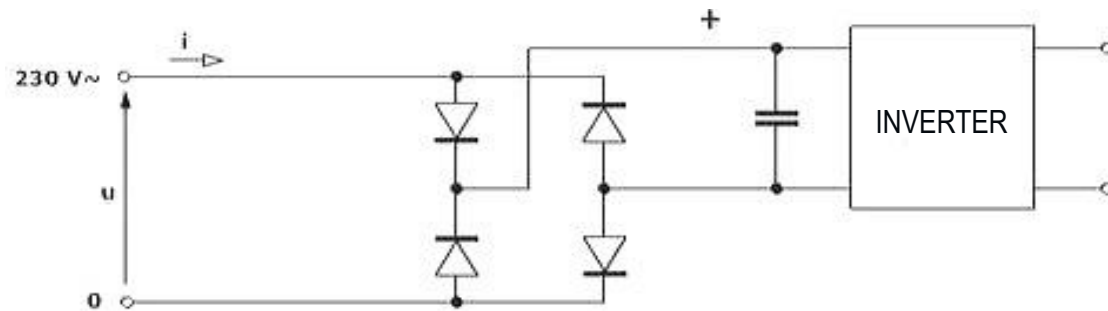
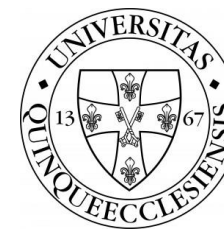
□ THD Factor

$$\rightarrow THD_V = \frac{\sqrt{\sum_{i=2}^{\infty} V_i^2}}{\sqrt{V_1^2}} = \frac{\sqrt{V^2 - V_1^2}}{V_1}$$

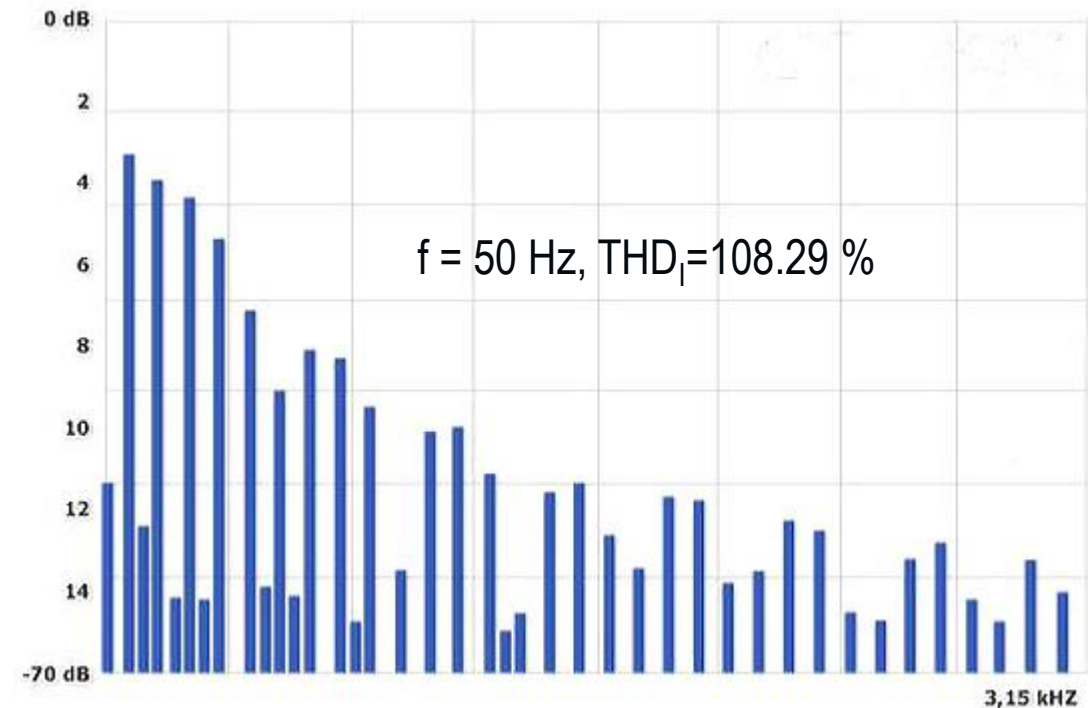
□ Klirr Factor

$$\rightarrow k_V = \frac{\sqrt{\sum_{i=2}^{\infty} V_i^2}}{\sqrt{\sum_{i=1}^{\infty} V_i^2}} = \frac{\sqrt{V^2 - V_1^2}}{V}$$

Power Quality (pf, THD)



- ❑ Switching PS, Electronic Devices → THD_i: 80-120%
- ❑ Negative affects - THD_i → THD_U (even only 3-4%)
 - ❑ Overheating of machines
 - ❑ Overheating of capacitor plants
 - ❑ Resonant overvoltages



Questions

