DR. GYURCSEK ISTVÁN

Multi-Wave Signals and Circuits

The Fourier Series

Sources and additional materials (recommended)

- *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- *Simonyi K.: Villamosságtan. AK Budapest 1983, ISBN:9630534134*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*
- *Zombory L.: Elektromágneses terek. MK Budapest 2006, (www.electro.uni-miskolc.hu)*

Progress in Content

- **Trigonometric Form of Fourier Series**
- Amplitude-Phase Form of Fourier Series
- Exponential Form of Fourier Series
- Circuit Applications
- Average Power and RMS Values

Trigonometric Fourier Series

$$f(t) = f(t + nT_0) \leftarrow n = 1, 2, 3, \ldots$$

Fourier's theorem – *Under suitable conditions* any periodic function can be represented by a Fourier series.

$$f(t) = A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t) \leftarrow \omega_0 = \frac{2\pi}{T_0}$$

Fourier series: $f(t)$ → DC component + AC component (infinite series of sinusoids).

Convergent *(requirements* → *Dirichlet* conditions; sufficient, not necessary)

- $f(t)$ is single-valued function everywhere.
- $f(t)$ has a finite number of finite discontinuities in one period.
- $f(t)$ has a finite number of maxima and minima in one period.
- The integral $\int_{t_0}^{t_0+T} |f(t)| dt < \infty$ for any t_0

***Note** – Theorem published by Fourier (1822), acceptable proof later by Dirichlet.*

Fourier Analysis

- ❑ Discrete Fourier analysis → process, determining the coefficients in Fourier series
- ❑ Continuous Fourier analysis → process, determining Fourier transform

Before we start …

$$\int_0^T \sin n\omega t \cos m\omega t \, dt = 0$$

$$\int_0^T \sin n\omega t \, dt = 0 \qquad \int_0^T \sin n\omega t \sin m\omega t \, dt = 0 \ \ (n \neq m), \qquad and \int_0^T \sin^2 n\omega t \, dt = \frac{T}{2}$$

$$\int_0^T \cos n\omega t \, dt = 0 \qquad \int_0^T \cos n\omega t \cos m\omega t \, dt = 0 \ \ (n \neq m), \qquad and \int_0^T \cos^2 n\omega t \, dt = \frac{T}{2}$$

Fourier Analysis $(A_0 =?,\ A_k =?,\ B_k =?)$

$$\int_0^T f(t)dt = \int_0^T A_0 dt + \sum_{k=1}^{\infty}\left(A_k \int_0^T \cos k\omega_0 t\, dt + B_k \int_0^T \sin k\omega_0 t\, dt\right)$$

$$\int_0^T f(t)dt = \int_0^T A_0 dt = A_0 T \qquad A_0 = \frac{1}{T}\int_0^T f(t)dt$$

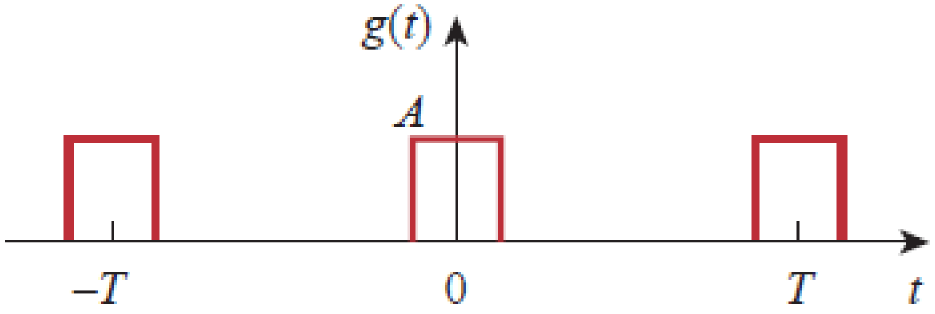
$$\int_0^T f(t)\cos m\omega_0 t\, dt = \int_0^T A_0 \cos m\omega_0 t\, dt + \sum_{k=1}^{\infty}\left(A_k \int_0^T \cos k\omega_0 t \cos m\omega_0 t\, dt + B_k \int_0^T \sin k\omega_0 t \cos m\omega_0 t\, dt\right)$$

$$\int_0^T f(t)\cos m\omega_0 t\, dt = A_k \frac{T}{2}, \qquad (k = m) \qquad A_k = \frac{2}{T}\int_0^T f(t)\cos k\omega_0 t\, dt$$

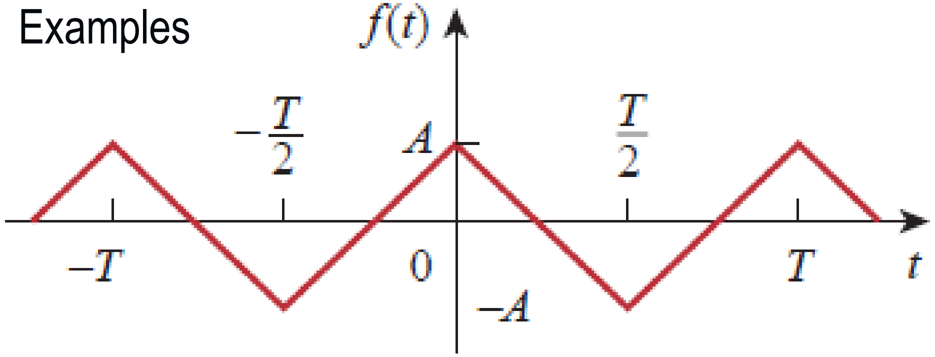
Homework → prove that $$B_k = \frac{2}{T}\int_0^T f(t)\sin k\omega_0 t\, dt$$

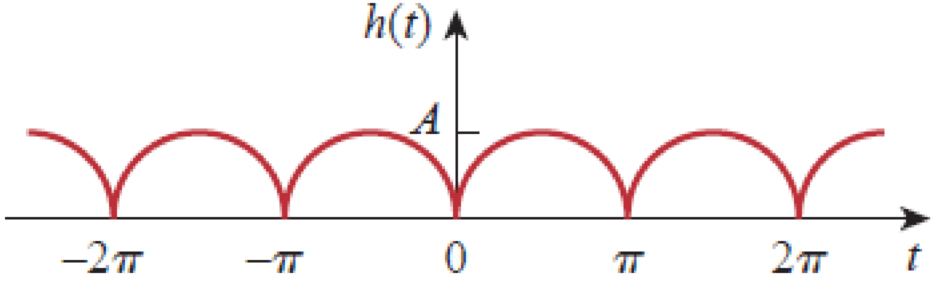
Symmetry Considerations 1

Even symmetry $f(t) = (f(-t)$



Examples

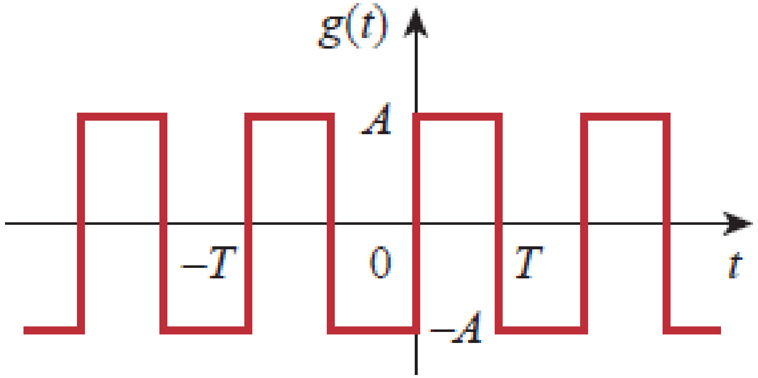




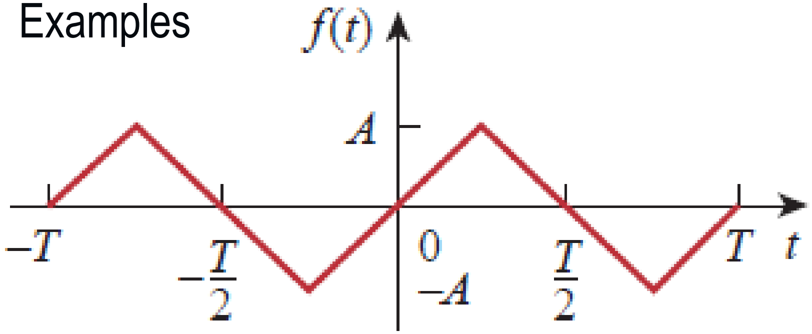
Main property $\rightarrow \int_{-T/2}^{T/2} f(t)dt = 2\int_{0}^{T/2} f(t)dt \rightarrow$ $A_0 = \frac{2}{T}\int_{0}^{T/2} f(t)dt$ $\quad A_k = \frac{4}{T}\int_{0}^{T/2} f(t)\cos k\omega_0 t\,dt$ $\quad B_k = 0$

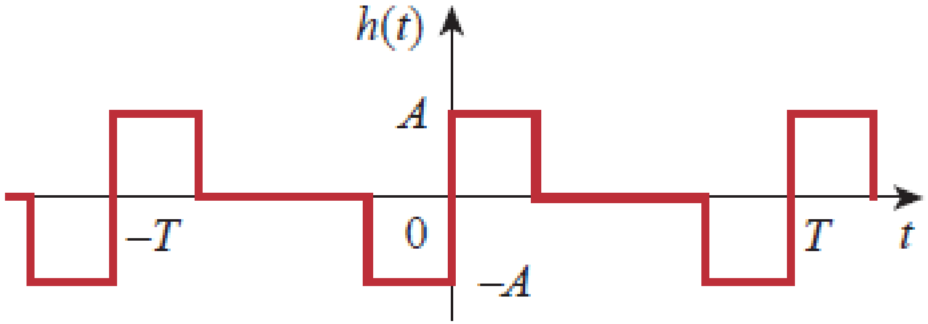
Symmetry Considerations 2

Odd symmetry $f(-t) = -(f(t)$



Examples



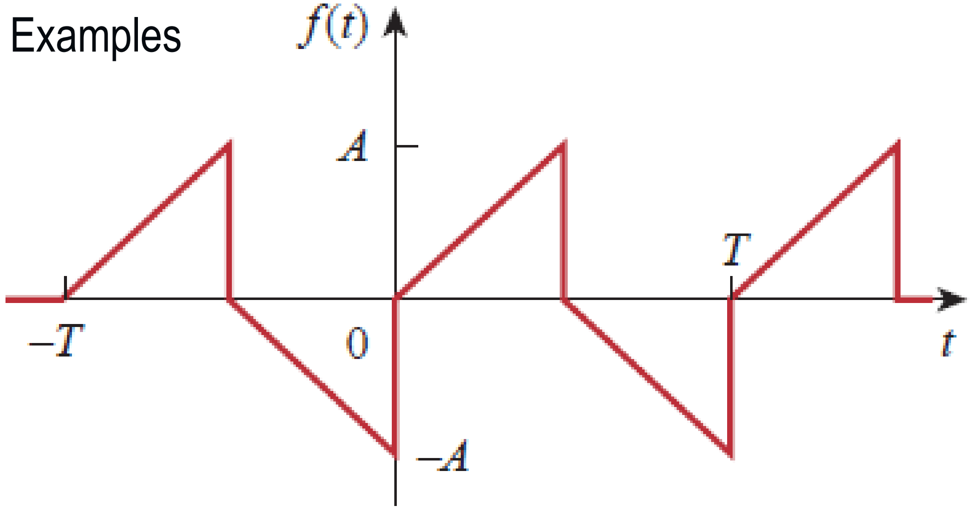


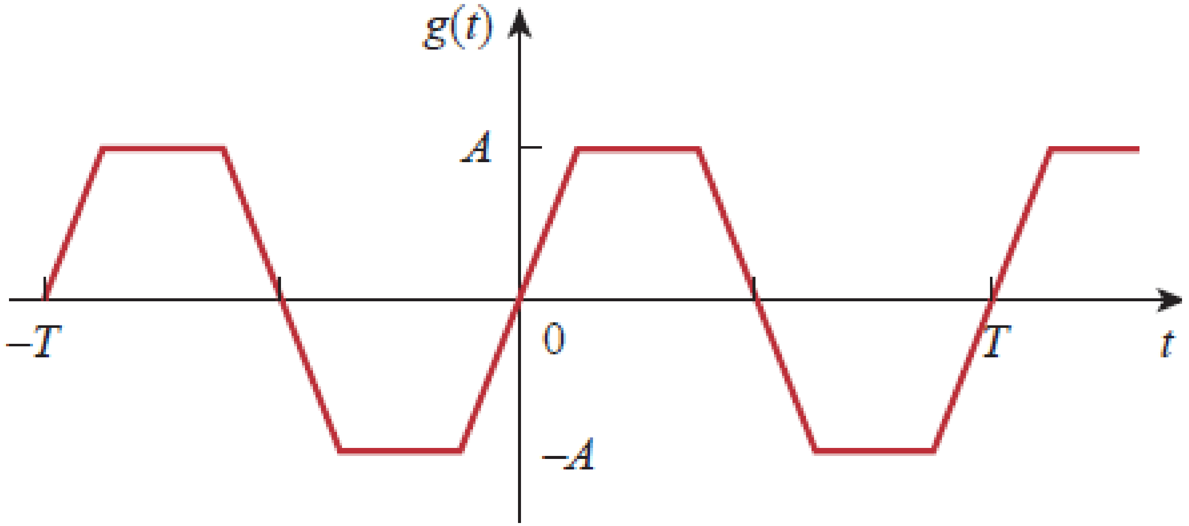
Main property $\rightarrow \int_{-T/2}^{T/2} f(t)dt = 0 \rightarrow$ $A_0 = 0$ $\qquad A_k = 0 \qquad B_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\omega_0 t\, dt$

Symmetry Considerations 3

Half-wave (odd) symmetry $f(t - T/2) = -(f(t)$ … next half-wave is the ‚mirror'

Examples





$$A_0 = 0 \qquad A_k = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos k\omega_0 t\,dt & k = odd \\ 0 & k = even \end{cases} \qquad B_k = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\sin k\omega_0 t\,dt & k = odd \\ 0 & k = even \end{cases}$$


Fourier

$$f(t) = f(t + nT)$$

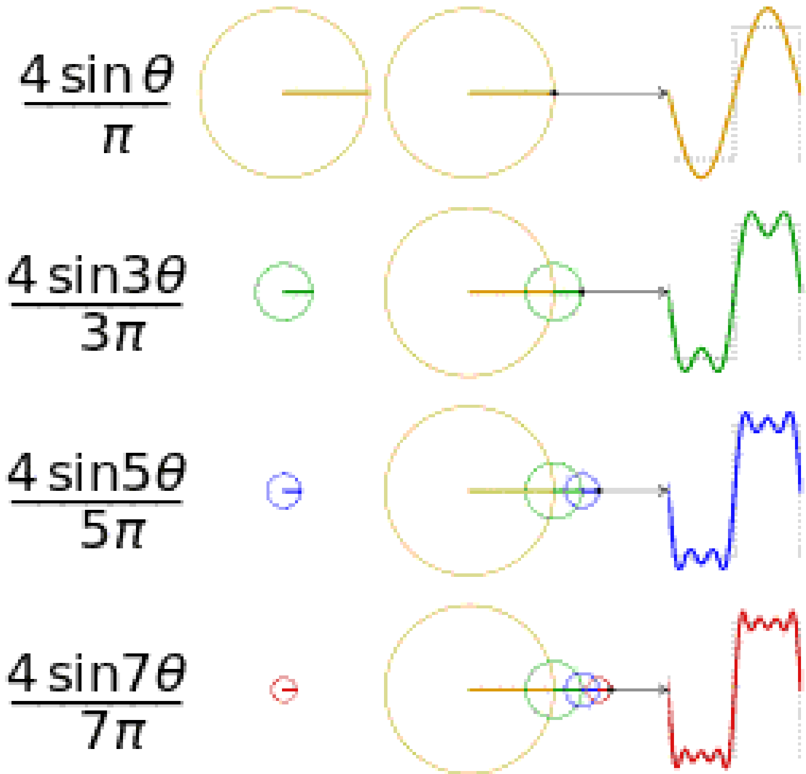
$$A_0 = \frac{1}{T}\int_0^T f(t)dt$$

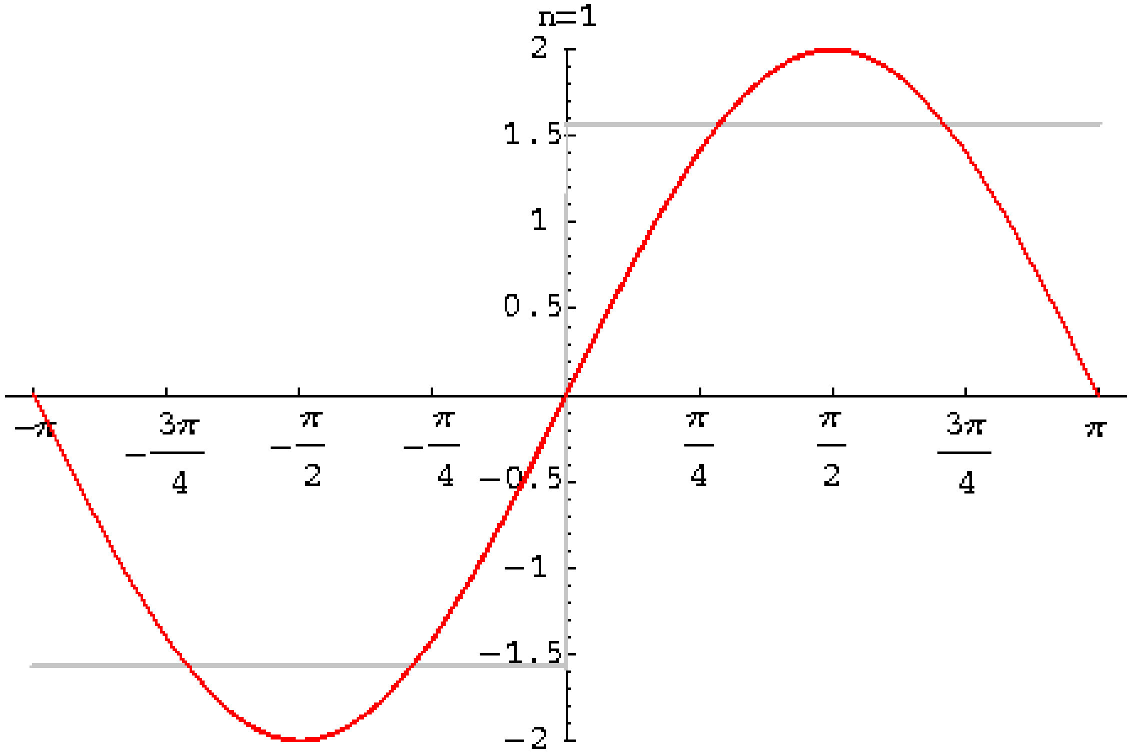
$$A_k = \frac{2}{T}\int_0^T f(t)\cos k\omega_0 t\, dt$$

$$f(t) = A_0 + \sum_{k=1}^{\infty}(A_k\, cos\, k\omega_0 t + B_k\, sin\, k\omega_0 t)$$

$$B_k = \frac{2}{T}\int_0^T f(t)\sin k\omega_0 t\, dt$$

Illustration – Square Wave Approach





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- ❑ **Amplitude-Phase Form of Fourier Series**
- ❑ Exponential Form of Fourier Series
- ❑ Circuit Applications
- ❑ Average Power and RMS Values

Amplitude-Phase Form of Fourier Series – Frequency Spectrum

$$f(t) = A_0 + \sum_{k=1}^{\infty} (A_k \, cos\, k\omega_0 t + B_k \, sin\, k\omega_0 t)$$

$$f(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t + \varphi_k)$$

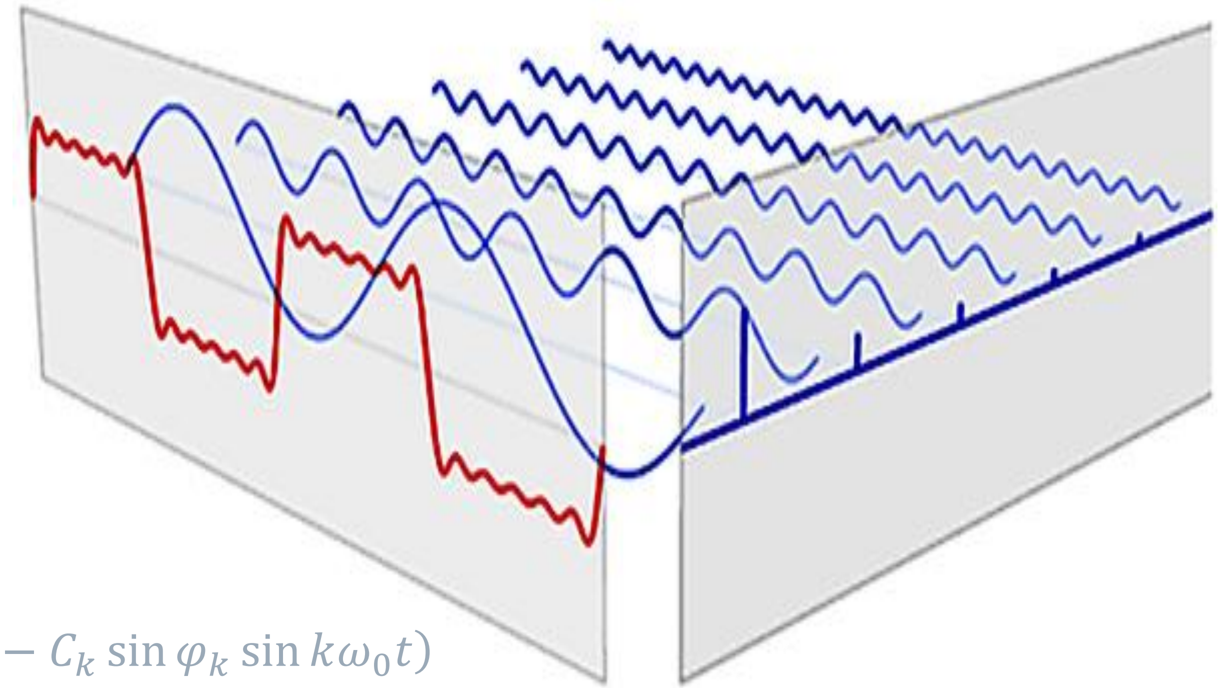
$$C_0 = A_0$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\sum_{k=1}^{\infty} C_k \cos(k\omega_0 t + \varphi_k) = \sum_{k=1}^{\infty} (C_k \cos\varphi_k \cos k\omega_0 t - C_k \sin\varphi_k \sin k\omega_0 t)$$

$$A_k = C_k \cos\varphi_k \, , \qquad B_k = -C_k \sin\varphi_k$$

$$C_k = \sqrt{A_k^2 + B_k^2}, \qquad \varphi_k = -tan^{-1}\frac{B_k}{A_k}$$



Frequency spectrum → plots

- ❑ amplitudes of the harmonics versus frequency
- ❑ phases of the harmonics versus frequency.

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Exponential Form of Fourier Series

$$\left.\begin{aligned} \cos k\omega_0 t &= \frac{e^{jk\omega_0 t} + e^{-jk\omega_0 t}}{2} \\ \sin k\omega_0 t &= \frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{j2} \end{aligned}\right\} \rightarrow f(t) = A_0 + \frac{1}{2}\sum_{k=1}^{\infty}\left[(A_k - jB_k)e^{jk\omega_0 t} + (A_k + jB_k)e^{-jk\omega_0 t}\right]$$

$$D_0 = A_0, \qquad \boldsymbol{D}_k = \frac{(A_k - jB_k)}{2}, \qquad \boldsymbol{D}_{-k} = \boldsymbol{D}_k^* = \frac{(A_k + jB_k)}{2} \rightarrow f(t) = D_0 + \sum_{k=1}^{\infty}\left(\boldsymbol{D}_k e^{jk\omega_0 t} + \boldsymbol{D}_{-k} e^{-jk\omega_0 t}\right)$$

$$f(t) = \sum_{k=-\infty}^{\infty} \boldsymbol{D}_k e^{jk\omega_0 t} \qquad \rightarrow \boldsymbol{D}_k = \frac{1}{T}\int_0^T f(t)\, e^{-jk\omega_0 t}\, dt$$

Exponential Fourier series of a periodic $f(t)$ describes the spectrum of $f(t)$
(amplitude and phase of ac components at positive and negative harmonic frequencies).

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Relationship between Three Forms

- Trigonometric Form of Fourier Series $$f(t) = A_0 + \sum_{k=1}^{\infty}(A_k\, cos\, k\omega_0 t + B_k\, sin\, k\omega_0 t)$$
- Amplitude-Phase Form of Fourier Series $$f(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t + \varphi_k)$$
- Exponential Form of Fourier Series $$f(t) = \sum_{k=-\infty}^{\infty} \boldsymbol{D}_k e^{jk\omega_0 t}$$

$$A_k = C_k \cos\varphi_k\,, \qquad B_k = -\,C_k \sin\varphi_k \qquad\qquad C_k = \sqrt{A_k^2 + B_k^2}, \qquad \varphi_k = -\,tan^{-1}\frac{B_k}{A_k}$$

$$\boldsymbol{D}_k = D_k\, e^{j\Theta_k} = \frac{(A_k - jB_k)}{2} = \frac{C_k}{2} e^{j\varphi_k} \rightarrow D_k = \frac{\sqrt{A_k^2 + B_k^2}}{2}, \qquad \Theta_k = -\,tan^{-1}\frac{B_k}{A_k} = \varphi_k$$

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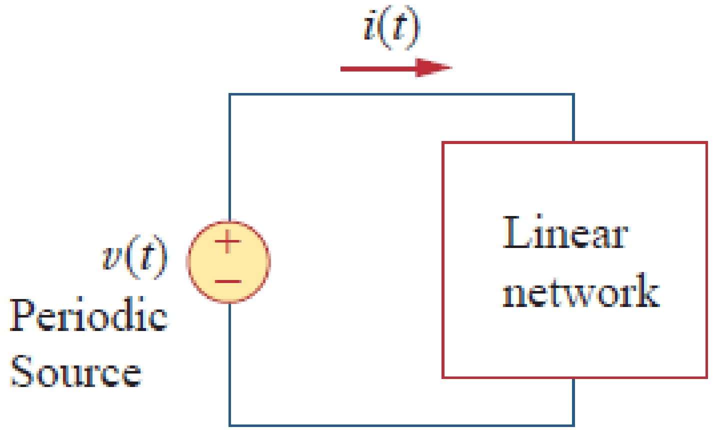
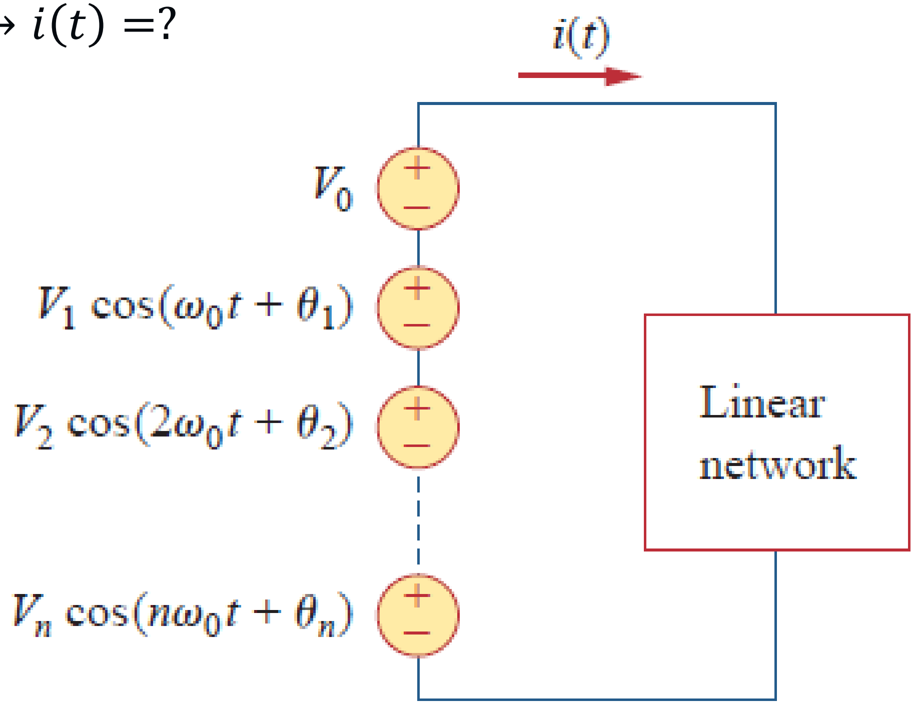
Circuit Application Procedure 1

Excitation $v(t)$ is a periodic source. Task → $i(t)$ =?

$$v(t) = V_0 + \sum_{k=1}^{\infty} V_k \cos(k\omega_0 t + \Theta_k)$$





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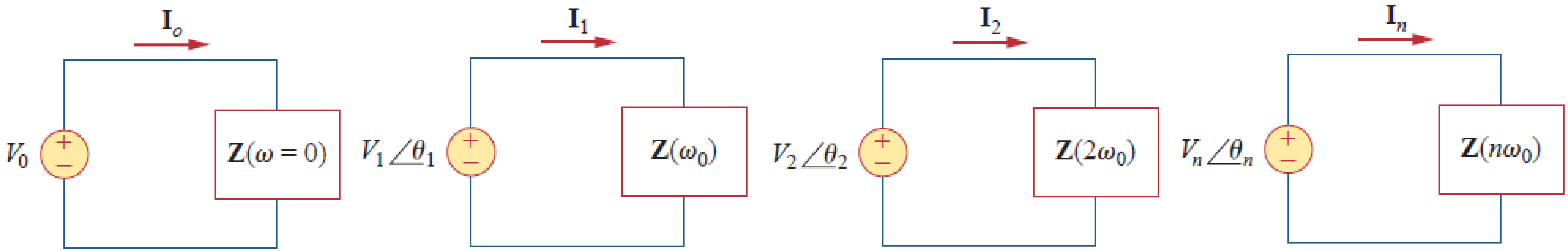
Circuit Application Procedure 2

(STEP 1) Express the excitation as a Fourier series

$$v(t) = V_0 + \sum_{k=1}^{\infty} V_k \cos(k\omega_0 t + \Theta_k)$$

(STEP 2) Find the response of each term in the series.

$$v_k(t) \rightarrow \boldsymbol{V}_\mathrm{k} \rightarrow \boldsymbol{I}_\mathrm{k} = \frac{\boldsymbol{V}_\mathrm{k}}{\boldsymbol{Z}_\mathrm{k}} \rightarrow i_k(t)\,, \qquad k = 0, 1, 2, \ldots$$



(STEP 3) Apply the superposition principle

$$i(t) = I_0 + i_1(t) + i_2(t) + \cdots = I_0 + \sum_{k=1}^{\infty} I_k \cos(k\omega_0 t + \varphi_k)$$

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Average Power

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \varphi_{Vn}) \qquad i(t) = I_0 + \sum_{m=1}^{\infty} I_m \cos(m\omega_0 t + \varphi_{Im}) \qquad P = \frac{1}{T}\int_0^T v(t)i(t)dt$$

$$P = \frac{1}{T}\int_0^T V_0 I_0 \mathrm{dt} + \sum_{m=1}^{\infty} \frac{V_0 I_m}{T}\int_0^T \cos(m\omega_0 t + \varphi_{Im})\, dt$$

zero !

$$+ \sum_{n=1}^{\infty} \frac{V_n I_0}{T}\int_0^T \cos(n\omega_0 t + \varphi_{Un})\, dt$$

zero!

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{V_n I_m}{T}\int_0^T cos(n\omega_0 t + \varphi_{Un})\, cos(m\omega_0 t + \varphi_{Im}) dt$$

n = m → non-zero

$$P = V_0 I_0 + \frac{1}{2}\sum_{n=1}^{\infty} V_n I_n \cos(\varphi_{Vn} - \varphi_{In})$$

$$P = \sum_{n=0}^{\infty} P_n$$

Total average power → Sum of the average powers in each harmonically related voltage and current.

RMS Values

$$F_{rms} = \sqrt{\frac{1}{T}\int_0^T f^2(t)\,dt}$$

$$F_{rms}^2 = \frac{1}{T}\int_0^T C_0^2\,dt + 2\sum_{n=1}^{\infty}\frac{C_0 C_n}{T}\int_0^T cos(n\omega_0 t + \varphi_n)\,dt + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{C_n C_m}{T}\int_0^T cos(n\omega_0 t + \varphi_n)\,cos(m\omega_0 t + \varphi_m)\,dt$$

$$F_{rms} = \sqrt{C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2}$$

$$F_{rms} = \sqrt{C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\sqrt{2}C_{rmsn}\right)^2} = \sqrt{C_0^2 + \sum_{n=1}^{\infty} C_{rmsn}^2}$$

$$F_{rms} = \sqrt{A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(A_n^2 + B_n^2)}$$

$$F_{rms}^2 = \sum_{n=0}^{\infty} F_{rmsn}^2$$

Parseval’s Theorem for Periodic Functions

$$F_{rms}^2 = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(A_n^2 + B_n^2)$$

Marc-Antoine Parseval
French mathematician (1755 - 1836)

$$f(t) = i(t) \rightarrow P = F_{rms}^2 R, \qquad f(t) = v(t) \rightarrow P = \frac{F_{rms}^2}{R}$$

To avoid nature of signal (current or voltage) $\rightarrow R = 1\,\Omega \rightarrow P_{1\,\Omega} = F_{rms}^2 = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(A_n^2 + B_n^2)$

Parseval’s theorem → average power in a periodic signal is the sum of

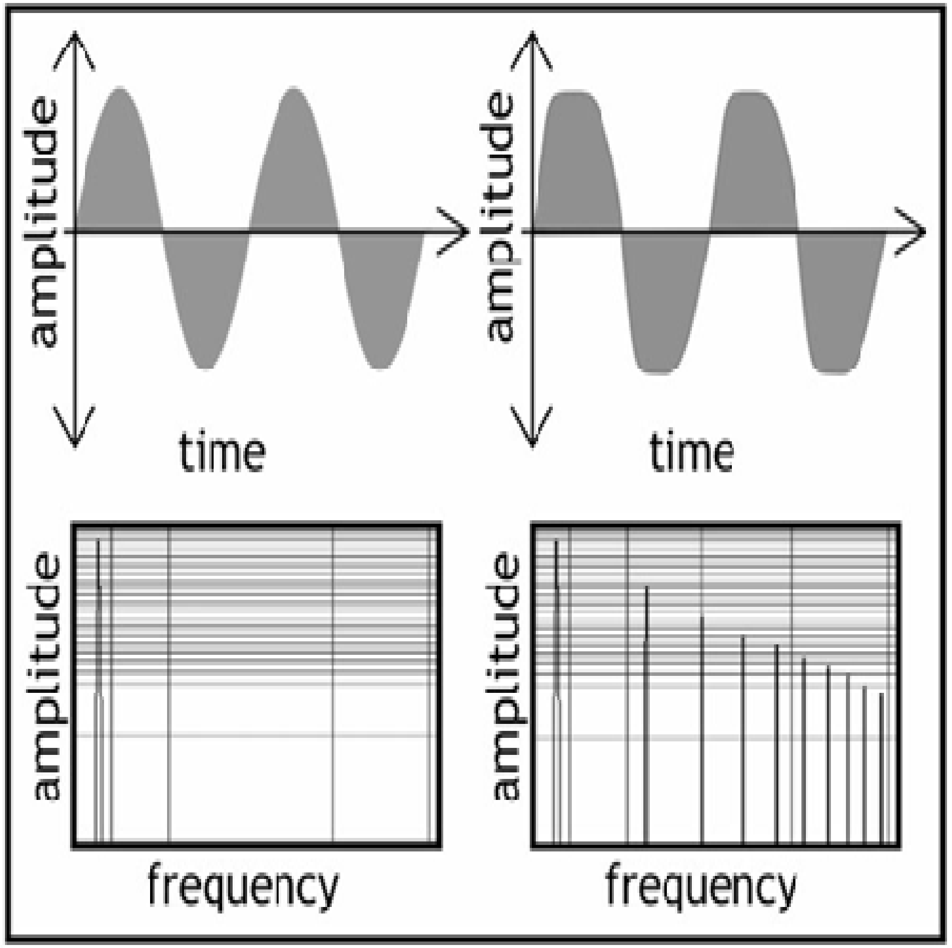
- ❑ average power in its dc component and
- ❑ average powers in its harmonics.

$$P = P_0 + \sum_{n=1}^{\infty} P_n$$

Parseval’s theorem (in general)

- ❑ Energy of periodic signals → concentrated at the harmonic components
- ❑ *Energy of nonperiodic signals → spread over the entire spectrum (Fourier transform … later on)*

RECALL: Specific Factors



- ❑ RMS Value → $V_{rms} = V = \sqrt{\frac{1}{T}\int_0^T v^2(t)\, dt}$
- ❑ D Factor → $D_{Vn} = \frac{V_n}{V_1}$
- ❑ THD Factor → $THD_V = \frac{\sqrt{\sum_{i=2}^{\infty} V_i^2}}{\sqrt{V_1^2}} = \frac{\sqrt{V^2 - V_1^2}}{V_1}$
- ❑ Klirr Factor → $k_V = \frac{\sqrt{\sum_{i=2}^{\infty} V_i^2}}{\sqrt{\sum_{i=1}^{\infty} V_i^2}} = \frac{\sqrt{V^2 - V_1^2}}{V}$

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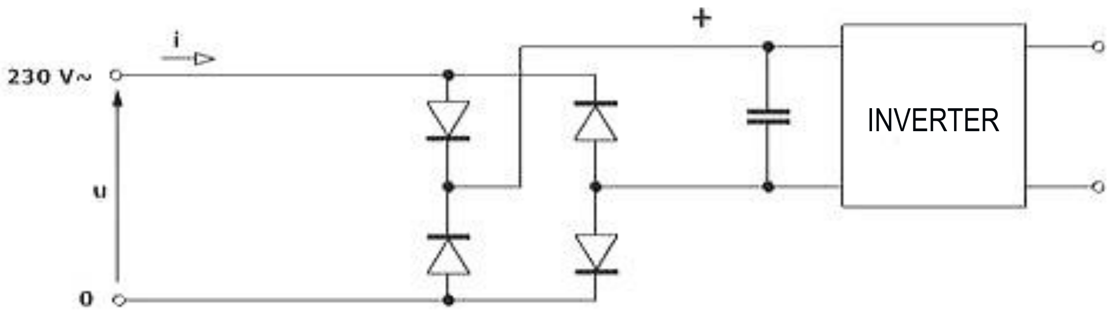
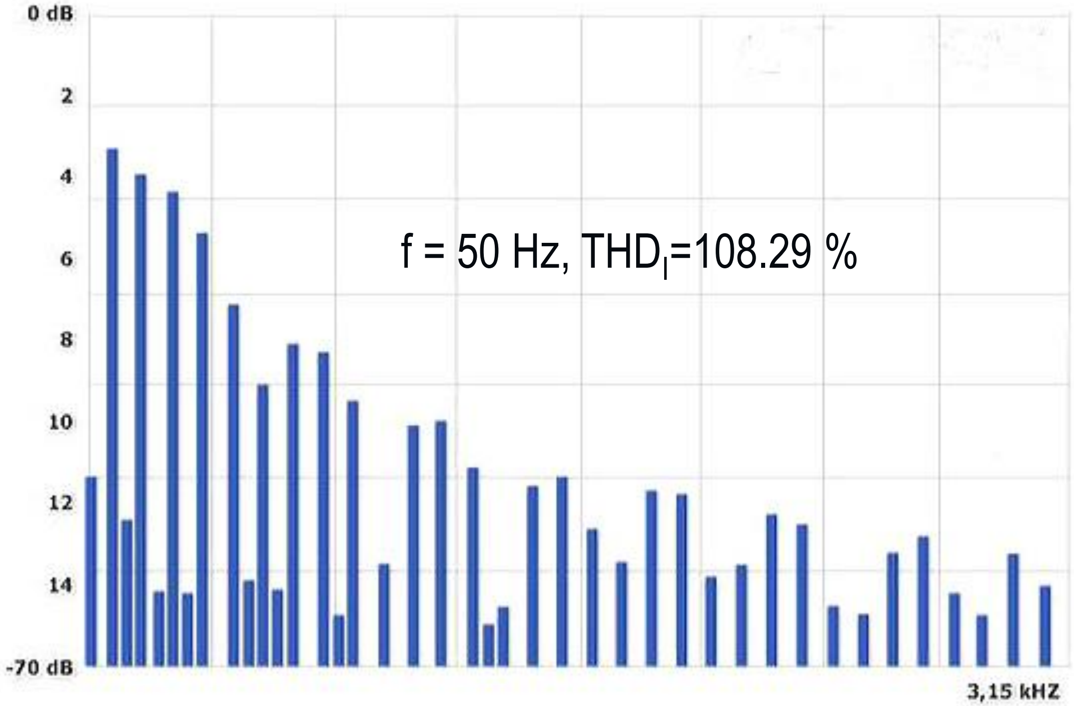
Power Quality (Example: pf, THD)





- Switching PS, Electronic Devices → THD$_I$: 80-120%
- Negative affects - THD$_I$ → THD$_U$ (even only 3-4%)
  - Overheating of machines
  - Overheating of capacitor plants
  - Resonant overvoltages

Questions