

DR. GYURCSEK ISTVÁN

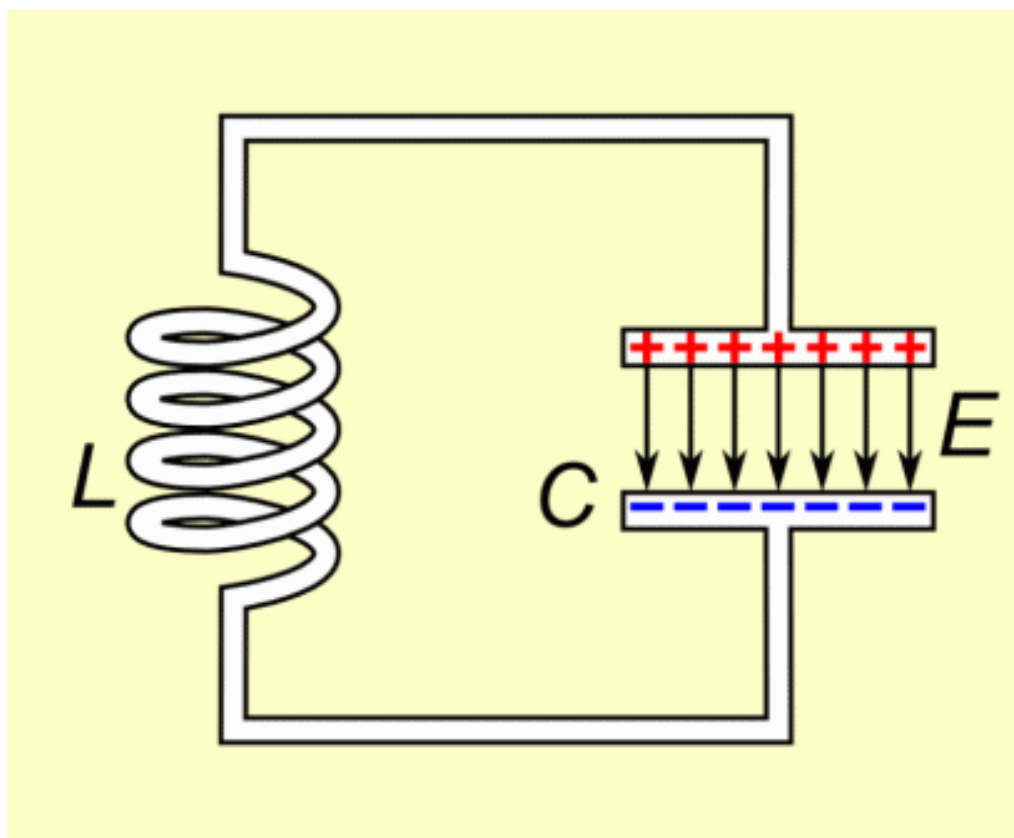
Dynamic Circuits 2

Second-Order Circuits

Sources and additional materials (recommended)

- ❑ *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- ❑ *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- ❑ *Prof. Townsend: Series RC, RL, and RLC Circuits, MTH 352 Fall 2005*
- ❑ *Parallel RC, RL, and RLC Circuits MTH 352 Fall 2005*
- ❑ *<http://www.ece.tufts.edu/~hopwood/downloads/es3/SecondOrderCircuits.ppt>*
- ❑ *<https://voer.edu.vn/c/second-order-circuits/24240886/072b46c4>*
- ❑ *Ormándlaky Zsolt: Átmeneti jelenségek*

Introduction (and reminder)



Static circuit

- COND1 - no storage element
- algebraic equations
- DC circuit*

Dynamic circuit

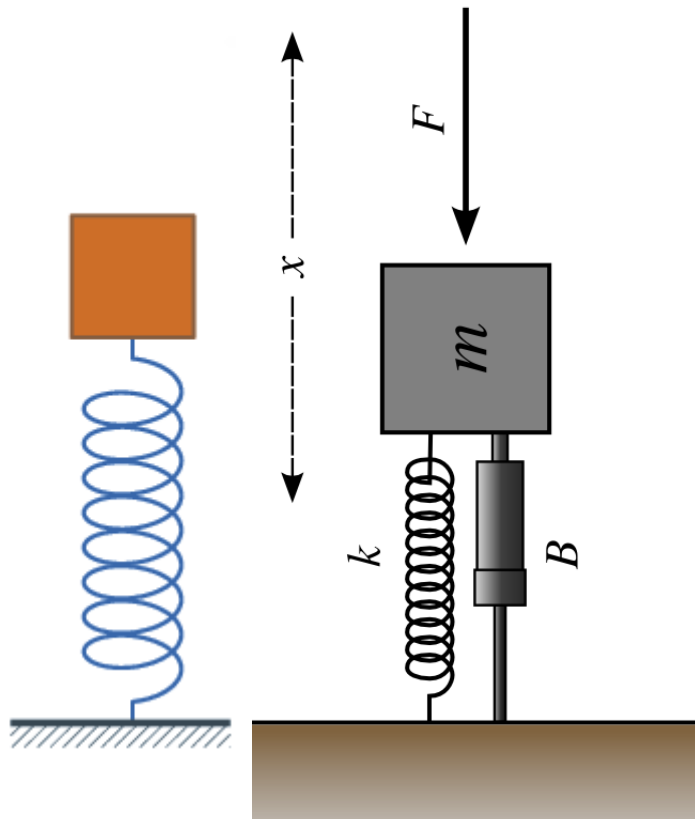
- COND1 – capacitor *AND/OR* inductor
- COND2 – time var. excitation *AND/OR* structure
- differential equations
- AC \rightarrow *part of DYN*

Transient Analysis 2

Second-order circuit

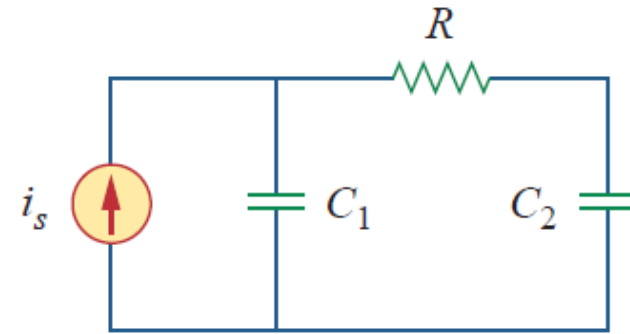
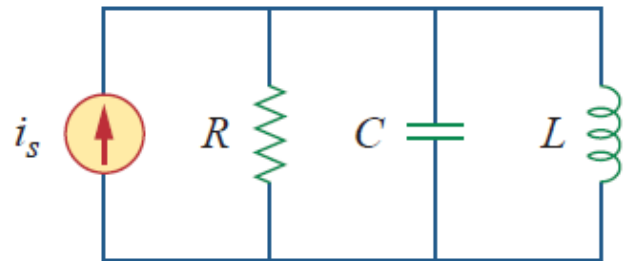
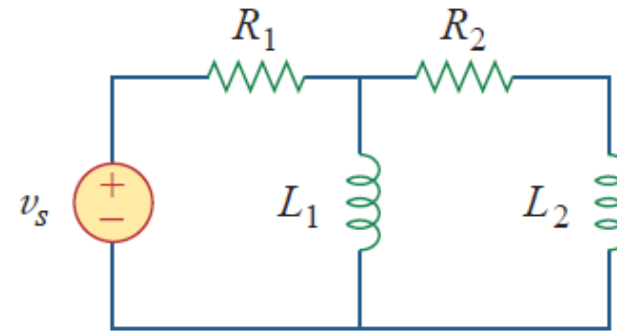
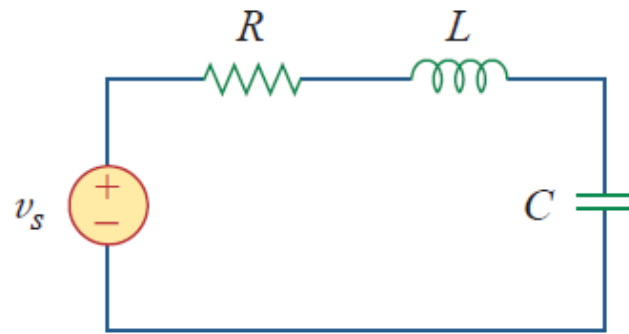
- Two independent (equivalent) reactive components*
- Characterized by 2nd-order differential equation*

Introduction – Mechanical Analogy



Mechanical Component	Electrical Component
Mass (m)	Inductor (L)
Spring (k)	Capacitor (C)
Shock absorber (B)	Resistor (R)
Force (F)	Voltage (V)
Velocity (dx/dt)	Current (dq/dt)

Introduction – Typical Second-Order Circuits

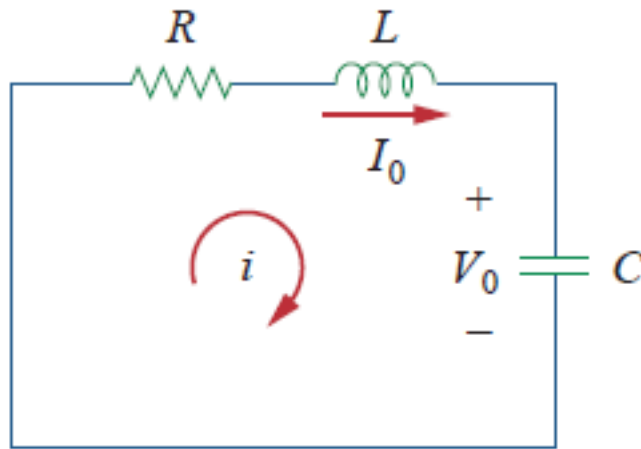




- The Source-Free Series RLC Circuit**
- Duality in Electric Circuits
- The Source-Free Parallel RLC Circuit
- Step Response of a Series RLC Circuit
- Step Response of a Parallel RLC Circuit
- General Second-Order Circuits

The Source-Free Series RLC Circuit

- Two separate 'legs' of initial energy → **two initial conditions**
- To solve second-order equation there must be two initial values



$$w_L(0) = \frac{1}{2} L i_L^2(0) \rightarrow i_L(0) = I_0$$

$$w_C(0) = \frac{1}{2} C v_C^2(0) \rightarrow v_C(0) = V_0$$

$$\frac{dv_C(0)}{dt} = \frac{1}{C} i(0) = \frac{I_0}{C}$$

$$v_L = L \frac{di}{dt}, \quad i = C \frac{dv_C}{dt}$$

$$v_L + v_R + v_C = 0 \quad v_L + RC \frac{dv_C}{dt} + v_C = 0$$

$$v_L = LC \frac{d^2 v_C}{dt^2} \quad LC \frac{d^2 v_C}{dt^2} + RC \frac{dv_C}{dt} + v_C = 0$$

'Complementary' solution

$$v_C(t) = K e^{st}$$

- K - from initial conditions
- s - from coefficients of diff.eq.

The Source-Free Series RLC Circuit

$$LC \frac{d^2 v_C}{dt^2} + R_T C \frac{dv_C}{dt} + v_C = 0$$

□ Complementary solution form

$$v_C(t) = K e^{st} \rightarrow K e^{st} (LC s^2 + R_T C s + 1) = 0$$

□ Characteristic equation

$$LC s^2 + R_T C s + 1 = 0$$

□ Quadratic characteristic eq. \rightarrow two roots

$$s_{1,2} = \frac{-R_T C \pm \sqrt{(R_T C)^2 - 4LC}}{2LC} = -\frac{R_T}{2L} \pm \sqrt{\left(\frac{R_T}{2L}\right)^2 - \frac{1}{LC}}$$

□ Introducing damping factor and natural frequency

$$\delta = \frac{R_T}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \rightarrow s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

□ Each root contributes in complementary solution

$$v_C(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

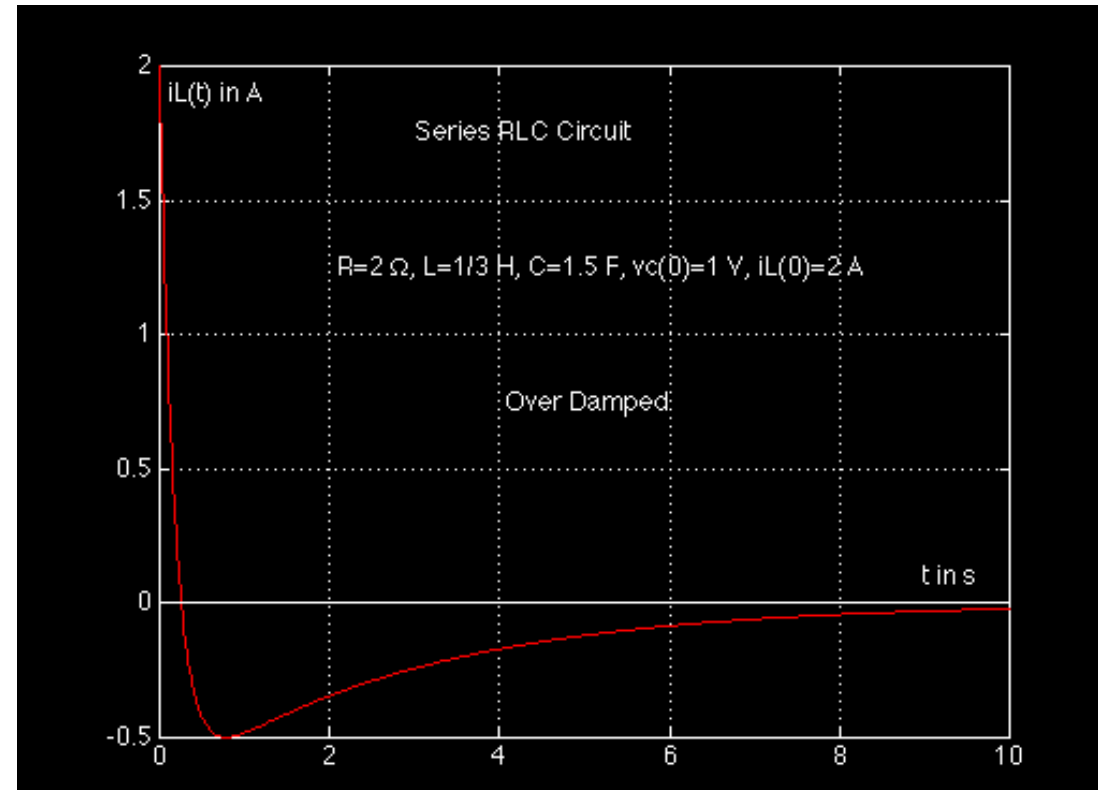
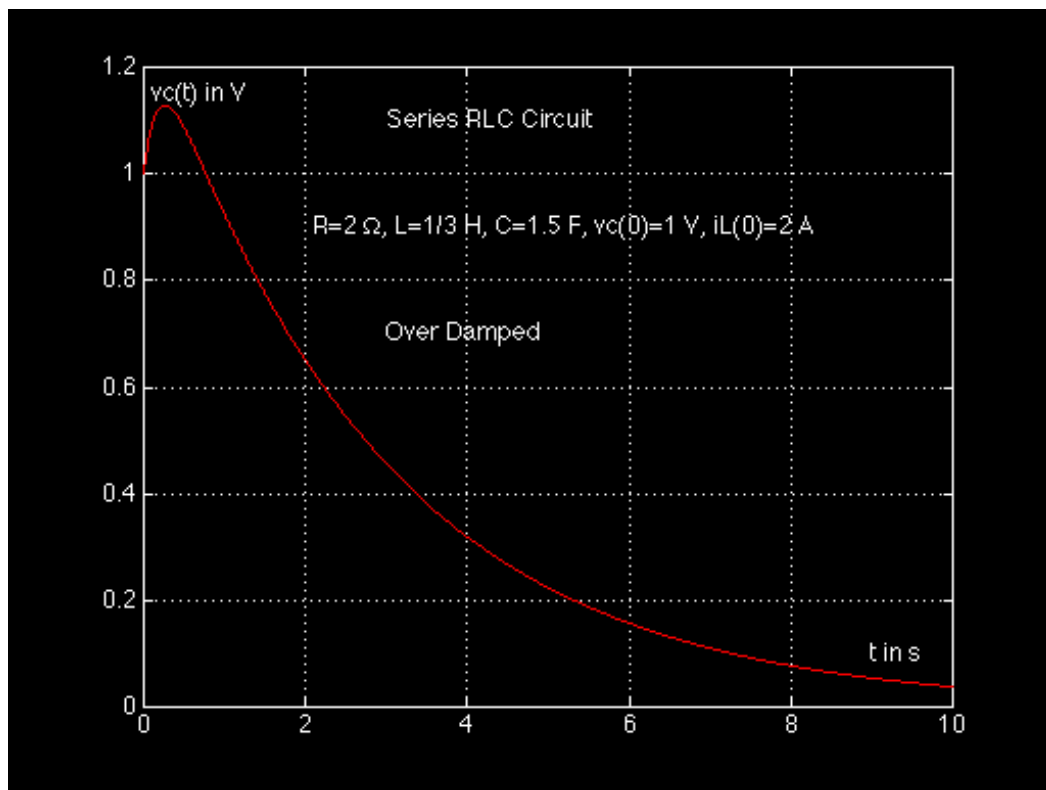
The Source-Free Series RLC Circuit

$$s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

CASE A	CASE B	CASE C	CASE D
$\delta^2 - \omega_0^2 > 0$	$\delta^2 - \omega_0^2 = 0$	$\delta^2 - \omega_0^2 < 0$	$\delta = 0$
$s_1 = -\delta + \sqrt{\delta^2 - \omega_0^2}$ $s_2 = -\delta - \sqrt{\delta^2 - \omega_0^2}$	$s_1 = s_2 = -\delta$	$s_1 = -\delta - j\sqrt{\omega_0^2 - \delta^2}$ $s_2 = -\delta + j\sqrt{\omega_0^2 - \delta^2}$	$s_1 = +j\omega_0$ $s_2 = -j\omega_0$
Two real unequal roots Over damped circuit	Two real equal roots Critically damped circuit	Two cpx. conjugate roots Under damped circuit	Two imaginary roots Undamped' circuit (free-oscillation)

Case A – Over Damped Circuit

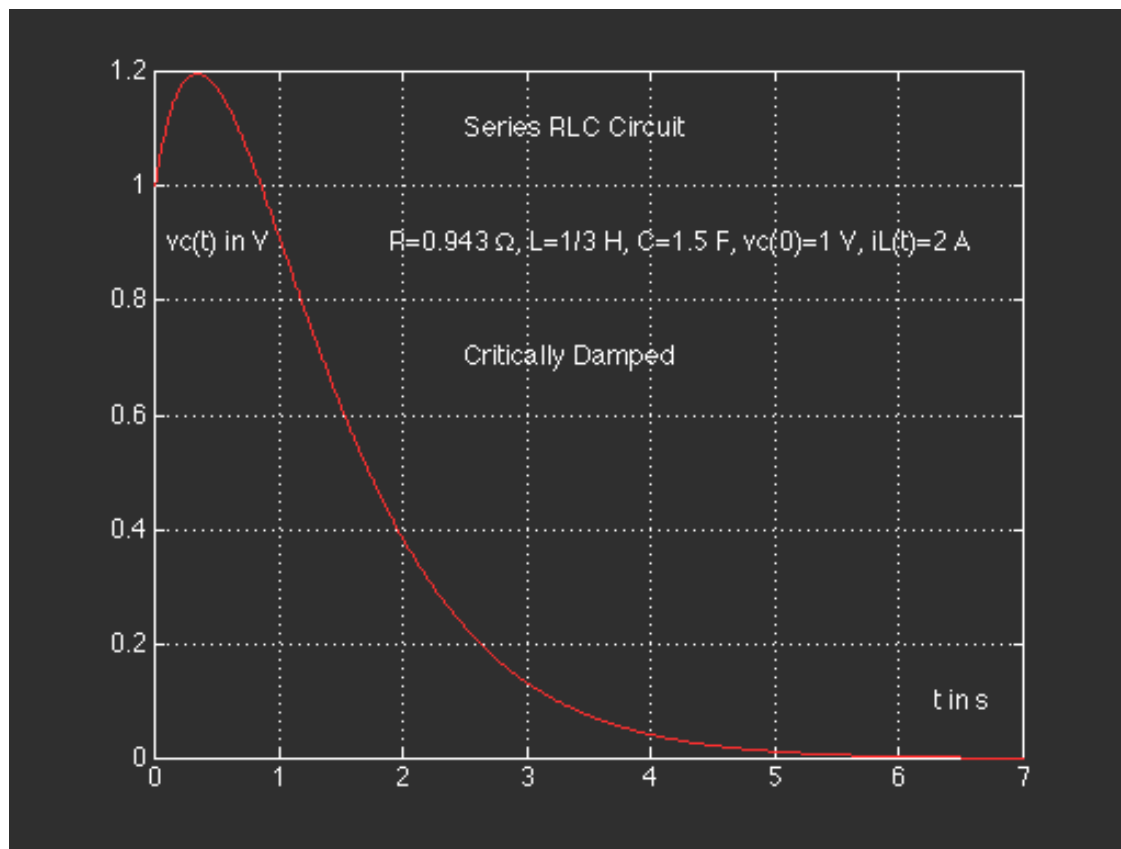
$$\delta^2 - \omega_0^2 > 0 \rightarrow s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} \rightarrow v_C(t) = K_1 e^{(-\delta + \sqrt{\delta^2 - \omega_0^2})t} + K_2 e^{(-\delta - \sqrt{\delta^2 - \omega_0^2})t}$$



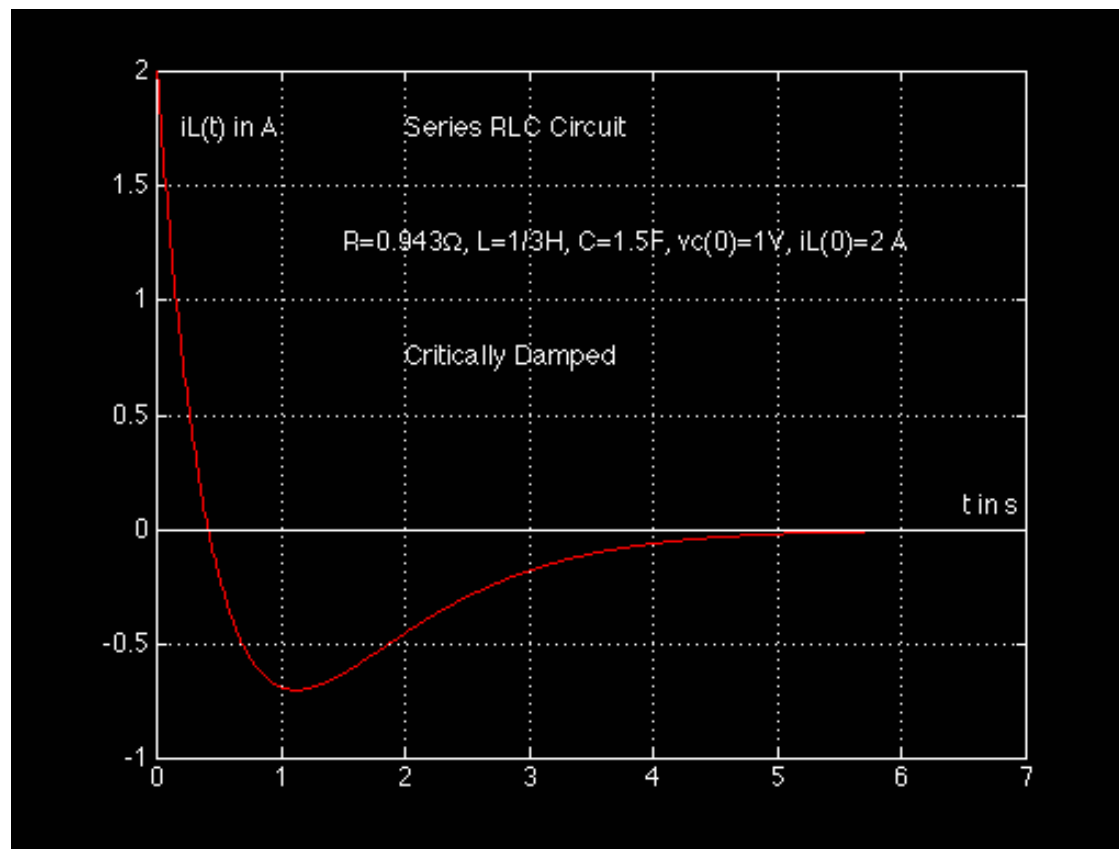
$$v_C(t) = (1.318e^{-0.35t} - 0.318e^{-5.646t}) \text{ V}, \quad i_L(t) = (-0.7e^{-0.35t} + 2.7e^{-5.646t}) \text{ A}$$

Case B – Critically Damped Circuit

$$\delta^2 - \omega_0^2 = 0 \rightarrow s_1 = s_2 = -\delta \rightarrow v_C(t) = K_1 e^{-\delta t} + K_2 t e^{-\delta t}$$



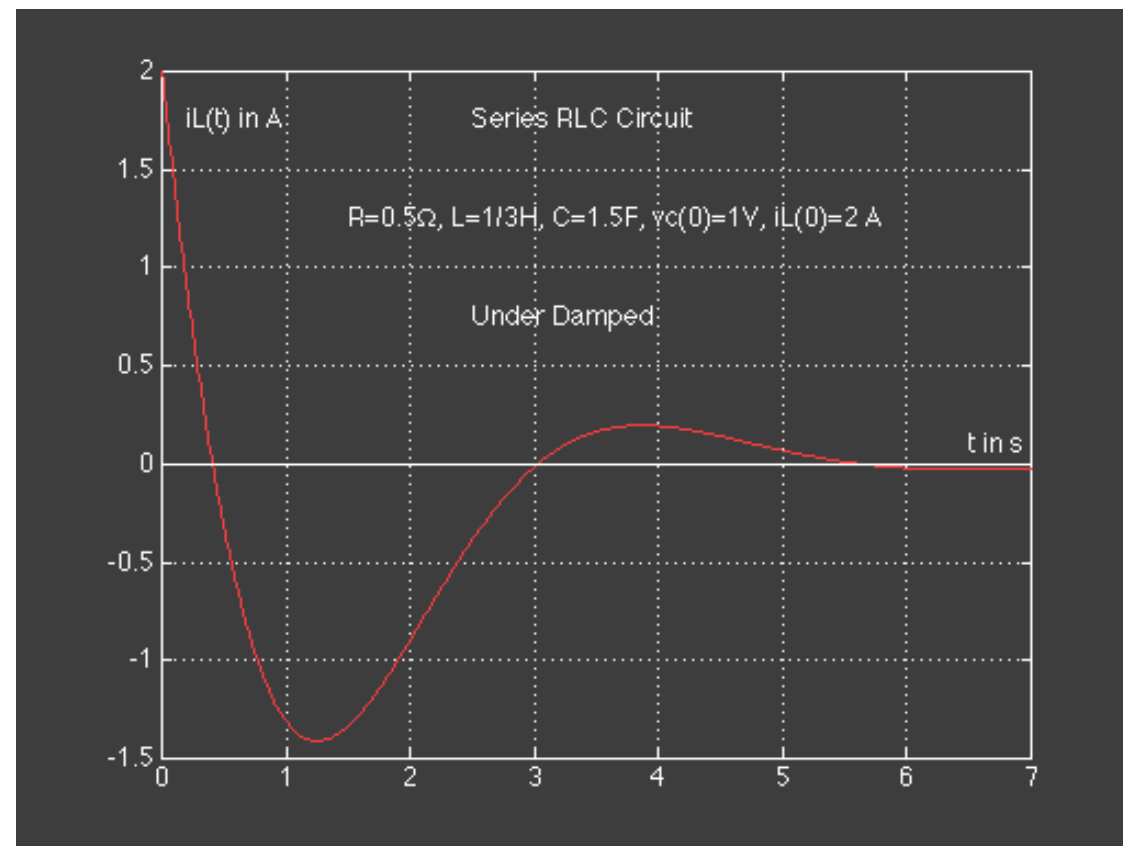
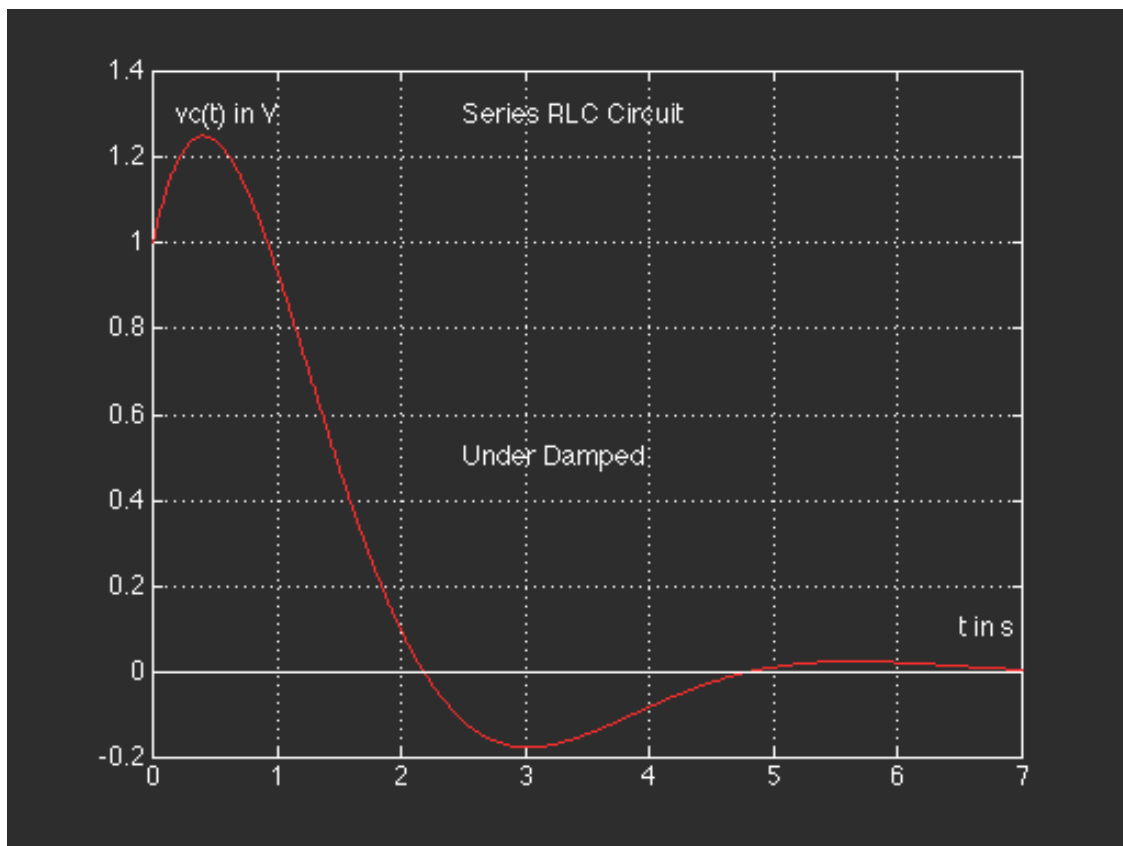
$$v_C(t) = (1.0e^{-1.414t} + 2.75te^{-1.414t}) V,$$



$$i_L(t) = (2e^{-1.414t} - 5.83te^{-1.414t}) A$$

Case C – Under Damped Circuit

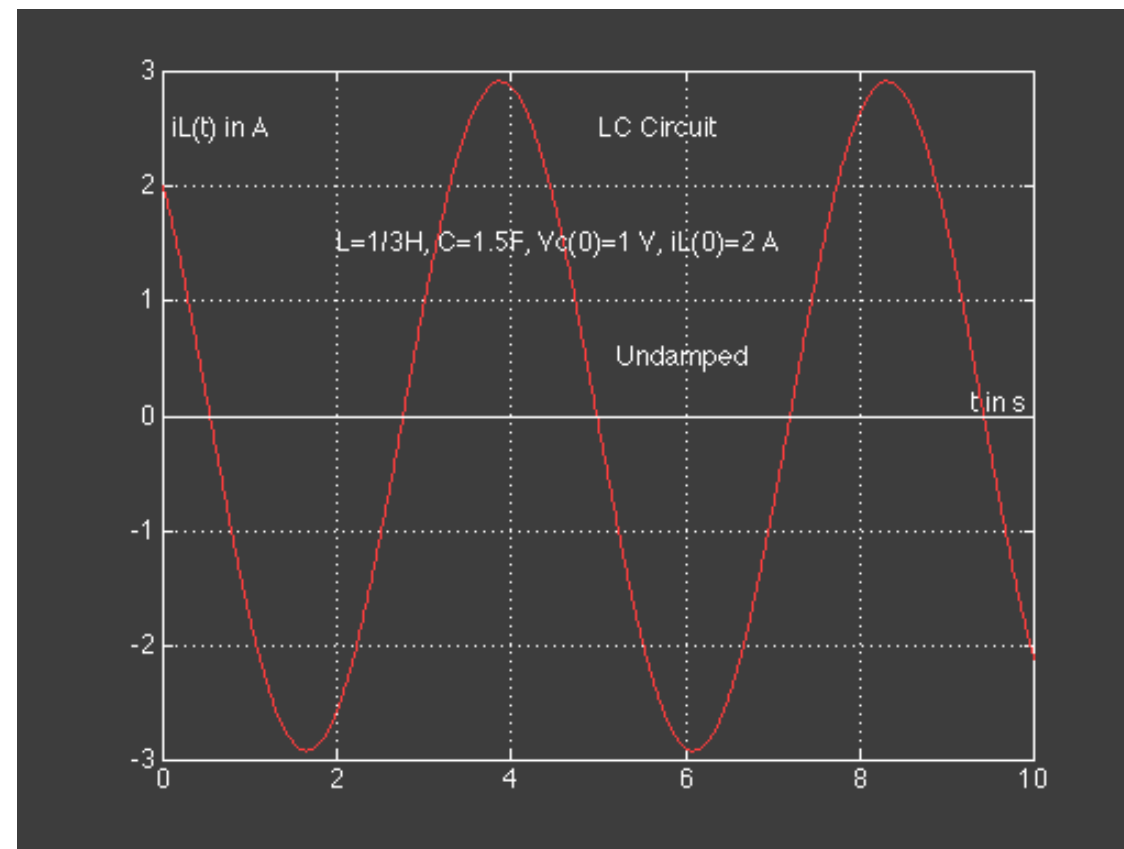
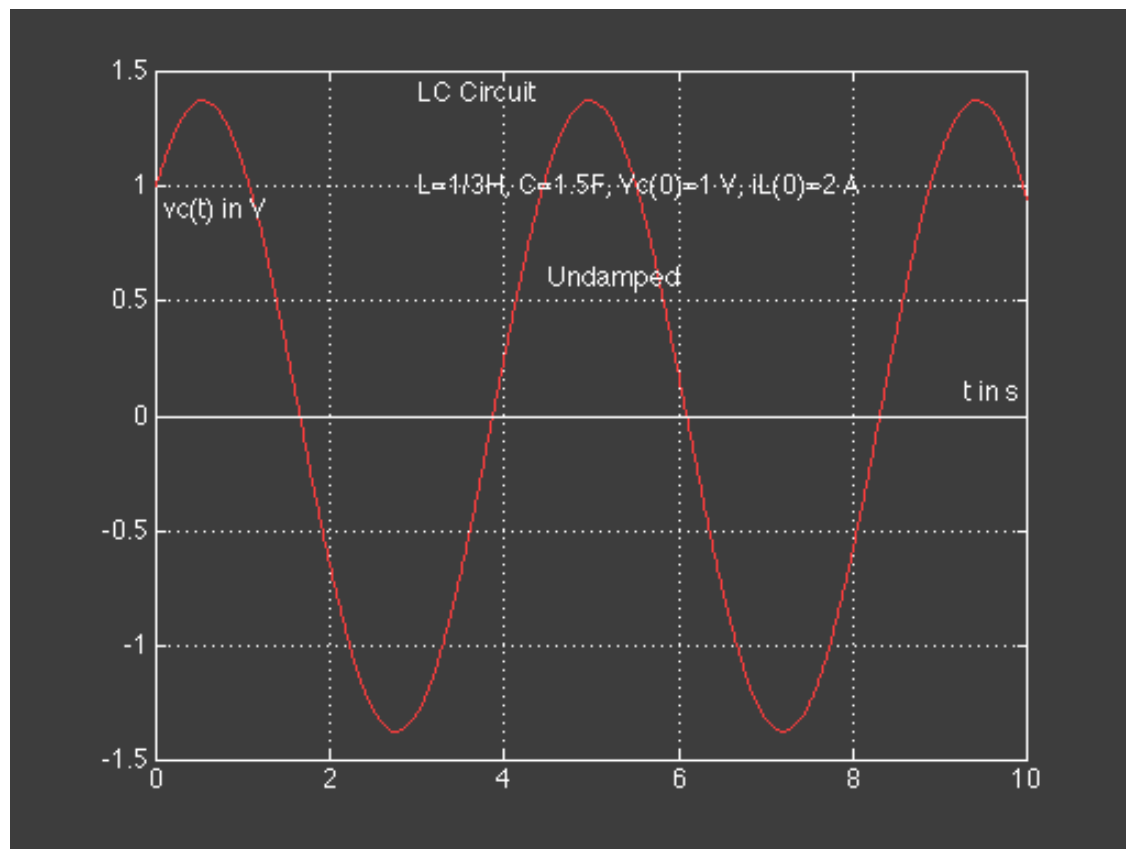
$$\delta^2 - \omega_0^2 < 0 \rightarrow s_{1,2} = -\delta \pm j\sqrt{\omega_0^2 - \delta^2} = -\delta \pm j\omega_d \rightarrow v_C(t) = e^{-\delta t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) = Ae^{-\delta t} \cos(\omega_d t + \varphi)$$



$$v_C(t) = 2e^{-0.75t} \cos(1.2t - 1.047) \text{ V}, \quad i_L(t) = 4.25e^{-0.75t} \cos(1.2t + 1.081) \text{ A}$$

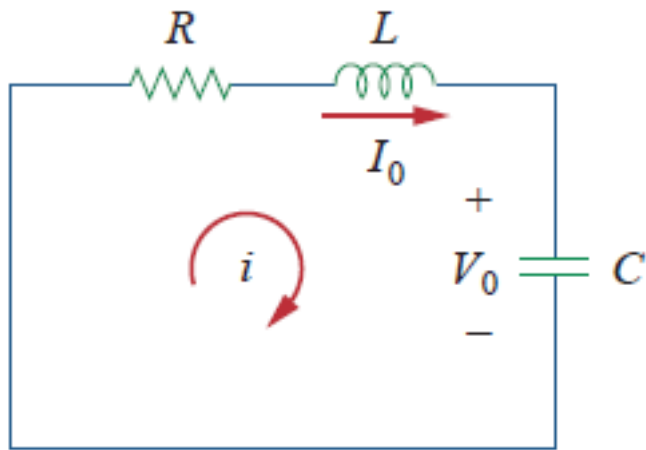
Case D – Undamped Circuit

$$\delta = 0 \rightarrow s_{1,2} = \pm j\omega_0 \rightarrow v_C(t) = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t = A \cos(\omega_0 t + \varphi)$$

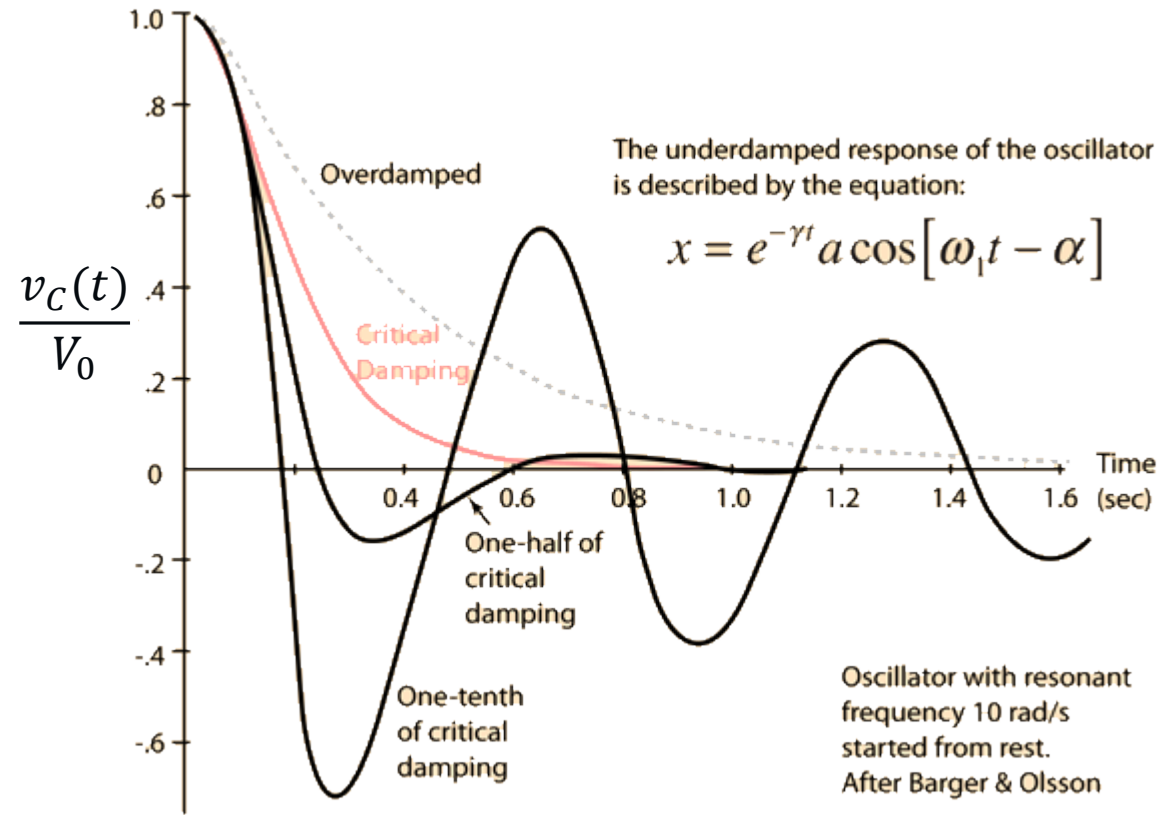


$$v_C(t) = 1.374 \cos(1.414t - 0.756) \text{ V}, \quad i_L(t) = 2.915 \cos(1.414t + 0.815) \text{ A}$$

The Source-Free Series RLC Circuit (Summary)



[undamped] vs. [damped]
 natural frequency; $\omega_0 \geq \omega_d$





- The Source-Free Series RLC Circuit
- Duality in Electric Circuits**
- The Source-Free Parallel RLC Circuit
- Step Response of a Series RLC Circuit
- Step Response of a Parallel RLC Circuit
- General Second-Order Circuits

Electrical Duality

Duality in circuit theory (A. Russell, 1904).

- Parallelism bw. pairs of characterizing equations and theorems of circuits.
- Interchanging dual params in an expression
- Dual expression has the same form

Examples

- Duality of electricity and magnetism*

$$\square \quad \sum v_i = 0 \quad \leftrightarrow \quad \sum i_i = 0$$

$$\square \quad \sum R_i = 0 \quad \leftrightarrow \quad \sum G_i = 0$$

$$\square \quad v_n = v_0 \frac{R_n}{\sum R_i} \quad \leftrightarrow \quad i_n = i_0 \frac{G_n}{\sum G_i}$$

$$\square \quad v_L = L \frac{di_L}{dt} \quad \leftrightarrow \quad i_C = C \frac{dv_C}{dt}$$

Dual Pair 1	Dual Pair 2
Resistance	Conductance
Inductance	Capacitance
Voltage	Current
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton



Dual circuits → same characterizing equations with interchanged dual quantities

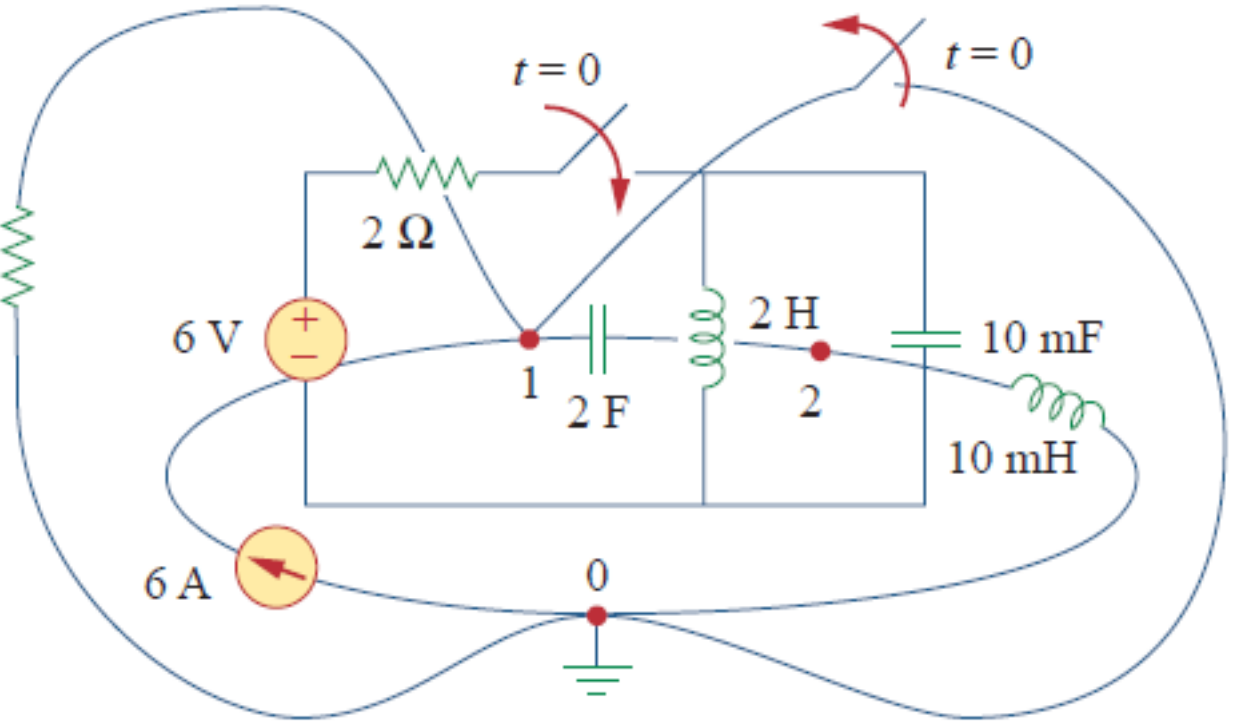
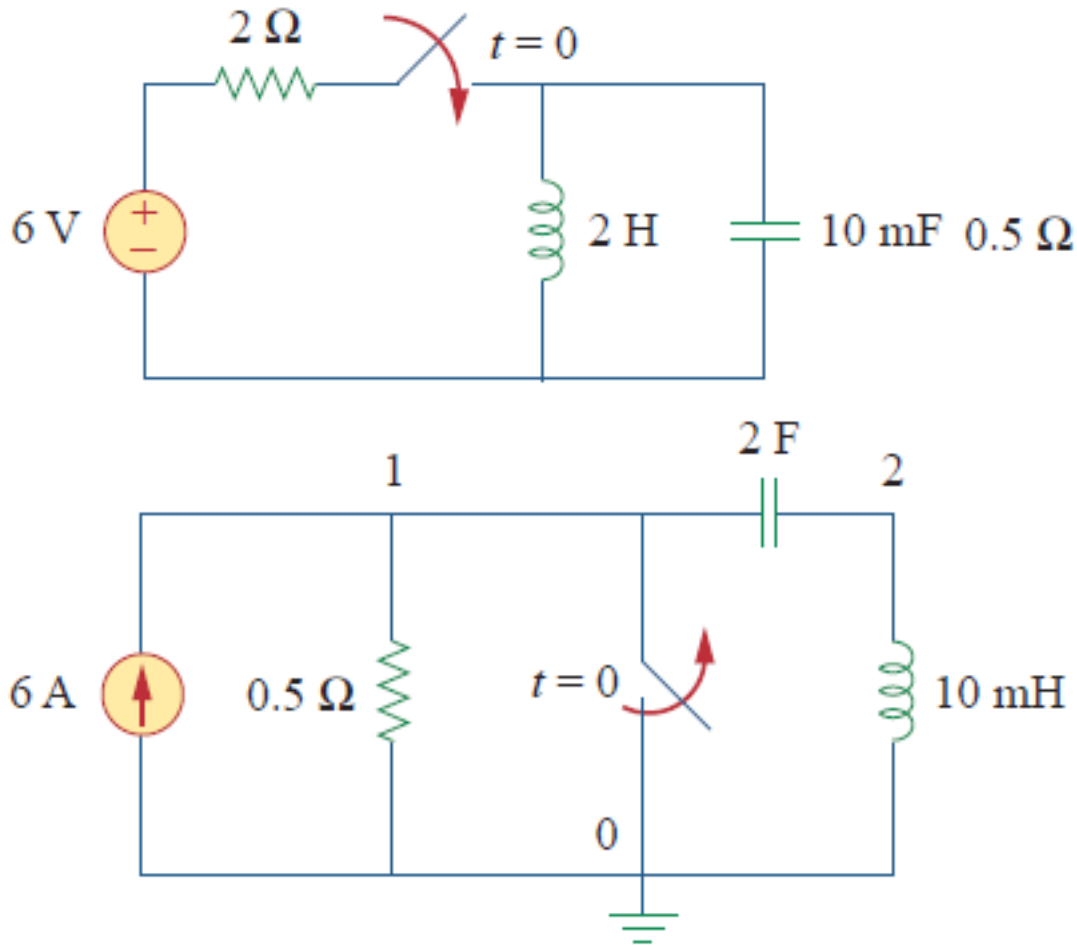
Notice - mutual inductance (for example) has no dual

Graphical technique to find the dual circuit

- Place a node at the center of each mesh of the given circuit.
Place the reference node (the ground) of the dual circuit outside the given circuit.
- Draw lines between the nodes such that each line crosses an element.
Replace that element by its dual.
- To determine the polarity of voltage sources and direction of current sources, follow this rule: A voltage source that produces a positive (clockwise) mesh current has as its dual a current source whose reference direction is from the ground to the nonreference node.

Electrical Duality - Example

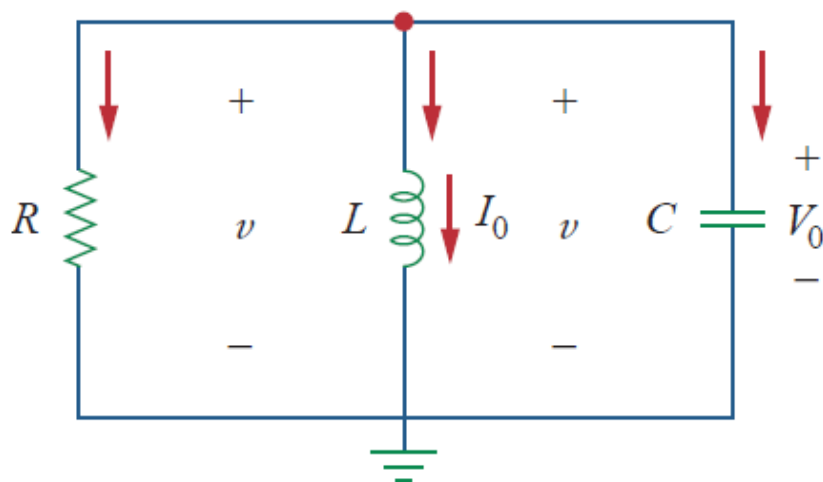
Find the dual circuit.





- The Source-Free Series RLC Circuit
- Duality in Electric Circuits
- The Source-Free Parallel RLC Circuit**
- Step Response of a Series RLC Circuit
- Step Response of a Parallel RLC Circuit
- General Second-Order Circuits

The Source-Free Parallel RLC Circuit



$$v = L \frac{di_L}{dt}, \quad i_C = C \frac{dv}{dt}$$

$$i_L + i_R + i_C = 0 \quad i_L + G_N L \frac{di_L}{dt} + i_C = i_N$$

$$i_C = LC \frac{d^2 i_L}{dt^2} \quad LC \frac{d^2 i_L}{dt^2} + GL \frac{di_L}{dt} + i_L = i_N$$

Initial conditions

To solve second-order equation there must be two initial values.

$$i_L(0) = I_0$$

$$\frac{di_L(0)}{dt} = \frac{1}{L} v(0) = \frac{V_0}{L}$$

'Complementary' solution

$$i_L(t) = K e^{st}$$

- K - from initial conditions
- S - from coefficients of diff.eq.

The Source-Free Parallel RLC Circuit

$$LC \frac{d^2 i_L}{dt^2} + GL \frac{di_L}{dt} + i_L = 0$$

□ Complementary solution form

$$i_L(t) = K e^{st} \rightarrow K e^{st} (LCs^2 + GLs + 1) = 0$$

□ Characteristic equation

$$LCs^2 + GLs + 1 = 0$$

□ Quadratic characteristic eq. \rightarrow two roots

$$s_{1,2} = \frac{-GL \pm \sqrt{(GL)^2 - 4LC}}{2LC} = -\frac{G}{2C} \pm \sqrt{\left(\frac{G}{2C}\right)^2 - \frac{1}{LC}}$$

□ Introducing damping factor and natural frequency

$$\delta = \frac{G}{2C}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \rightarrow s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

□ Each root contributes in complementary solution

$$i_L(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

The Source-Free Parallel RLC Circuit

$$s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

CASE A	CASE B	CASE C	CASE D
$\delta^2 - \omega_0^2 > 0$	$\delta^2 - \omega_0^2 = 0$	$\delta^2 - \omega_0^2 < 0$	$\delta = 0$
$s_1 = -\delta + \sqrt{\delta^2 - \omega_0^2}$ $s_2 = -\delta - \sqrt{\delta^2 - \omega_0^2}$	$s_1 = s_2 = -\delta$	$s_1 = -\delta - j\sqrt{\omega_0^2 - \delta^2}$ $s_2 = -\delta + j\sqrt{\omega_0^2 - \delta^2}$	$s_1 = +j\omega_0$ $s_2 = -j\omega_0$
Two real unequal roots Over damped circuit	Two real equal roots Critically damped circuit	Two cpx. conjugate roots Under damped circuit	Two imaginary roots Undamped' circuit (free-oscillation)

The Source-Free Parallel RLC Circuit

Case A (over damped)

$$\delta^2 - \omega_0^2 > 0 \rightarrow s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} \rightarrow i_L(t) = K_1 e^{(-\delta + \sqrt{\delta^2 - \omega_0^2})t} + K_2 e^{(-\delta - \sqrt{\delta^2 - \omega_0^2})t}$$

Case B (critically damped)

$$\delta^2 - \omega_0^2 = 0 \rightarrow s_1 = s_2 = -\delta \rightarrow i_L(t) = K_1 e^{-\delta t} + K_2 t e^{-\delta t}, \quad t \geq 0$$

Case C (under damped)

$$\delta^2 - \omega_0^2 < 0 \rightarrow s_{1,2} = -\delta \pm j\sqrt{\omega_0^2 - \delta^2} = -\delta \pm j\omega_d \rightarrow i_L(t) = e^{-\delta t} (A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

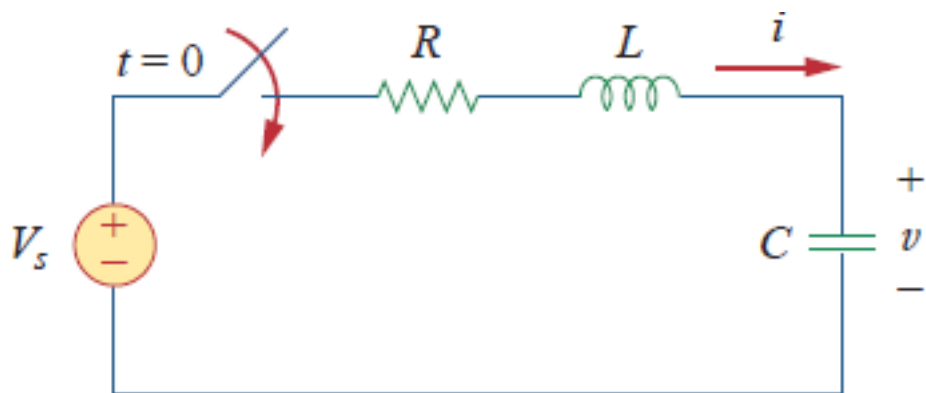
Case D (undamped)

$$\delta = 0 \rightarrow s_{1,2} = \pm j\omega_0 \rightarrow i_L(t) = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t$$



- The Source-Free Series RLC Circuit
- Duality in Electric Circuits
- The Source-Free Parallel RLC Circuit
- Step Response of a Series RLC Circuit**
- Step Response of a Parallel RLC Circuit
- General Second-Order Circuits

Step Response of a Series RLC Circuit



$$L \frac{di}{dt} + Ri + v = V_S, \quad i = C \frac{dv}{dt}$$

$$\frac{d^2v}{dt^2} + \frac{R}{L} \frac{dv}{dt} + \frac{v}{LC} = \frac{V_S}{LC}$$

Complete response = **transient response** ('temporary part, dies out w. time') + **steady-state response** (',permanent part')

$$v(t) = v_{tr}(t) + v_{ss}(t)$$

Transient part

- Overdamped $v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$
- Critically damped $v(t) = (K_1 + K_2) e^{-\delta t}$
- Underdamped $v(t) = (K_1 \cos \omega_d t + K_2 \sin \omega_d t) e^{-\delta t}$

Steady-state part $v_{ss}(t) = v(\infty) = V_S$

In general (for any x variable)

$$x(t) = x_{tr}(t) + x_{ss}(t)$$

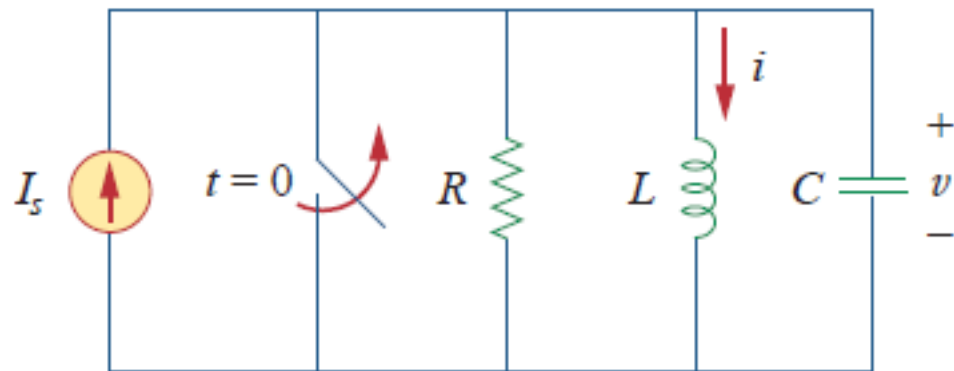
$$K_1, K_2 \leftarrow x(0), \frac{dx(0)}{dt} \text{ initial conditions}$$

$$s_1, s_2 \dots (\delta, \omega_d) \leftarrow RLC \text{ circuit elements}$$



- The Source-Free Series RLC Circuit
- Duality in Electric Circuits
- The Source-Free Parallel RLC Circuit
- Step Response of a Series RLC Circuit
- Step Response of a Parallel RLC Circuit**
- General Second-Order Circuits

Step Response of a Parallel RLC Circuit



$$C \frac{dv}{dt} + i + \frac{v}{R} = I_s, \quad v = L \frac{di}{dt}$$

$$\frac{d^2i}{dt^2} + \frac{1}{RC} \frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

Complete response = **steady-state response** (‘permanent part’) + **transient response** (‘temporary part, dies out w. time’)

$$i(t) = i_{ss}(t) + i_{tr}(t)$$

Complete response

- Overdamped $i(t) = I_s + K_1 e^{s_1 t} + K_2 e^{s_2 t}$
- Critically damped $i(t) = I_s + (K_1 + K_2) e^{-\delta t}$
- Underdamped $i(t) = I_s + (K_1 \cos \omega_d t + K_2 \sin \omega_d t) e^{-\delta t}$

In general (for any x variable)

$$x(t) = x_{tr}(t) + x_{ss}(t)$$

$$K_1, K_2 \leftarrow x(0), \frac{dx(0)}{dt} \text{ initial conditions}$$

$$s_1, s_2 \dots (\delta, \omega_d) \leftarrow RLC \text{ circuit elements}$$



- The Source-Free Series RLC Circuit
- Duality in Electric Circuits
- The Source-Free Parallel RLC Circuit
- Step Response of a Series RLC Circuit
- Step Response of a Parallel RLC Circuit
- General Second-Order Circuits**

General Second-Order Circuits

- ❑ Find the initial conditions $x(0)$, $dx(0)/dt$ and the final value $x(\infty)$
- ❑ Turn off the independent sources. Find $x_{tr}(t)$ by KCL/KVL (2nd-order diff. eq.
 → characteristic roots (over-, critically, or underdamped) → $x_{tr}(t)$ w. 2 unknown constants
- ❑ Obtain the steady-state response $x_{ss}(t) = x(\infty)$
- ❑ Total response $x(t) = x_{tr}(t) + x_{ss}(t)$
- ❑ Determine the constants in transient response by $x(0)$, $dx(0)/dt$.

Check your skill.

$v_0(t) = ?$ for $t > 0$.
 (Hint: first find v_1 and v_2 .)

Result $8(e^{-t} - e^{-6t})$ V

