DR. GYURCSEK ISTVÁN

Dynamic Circuits 2

Second-Order Circuits

Sources and additional materials (recommended)

- *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- *Prof. Townsend: Series RC, RL, and RLC Circuits, MTH 352 Fall 2005*
- *Parallel RC, RL, and RLC Circuits MTH 352 Fall 2005*
- *http://www.ece.tufts.edu/~hopwood/downloads/es3/SecondOrderCircuits.ppt*
- *https://voer.edu.vn/c/second-order-circuits/24240886/072b46c4*
- *Ormándlaky Zsolt: Átmeneti jelenségek*

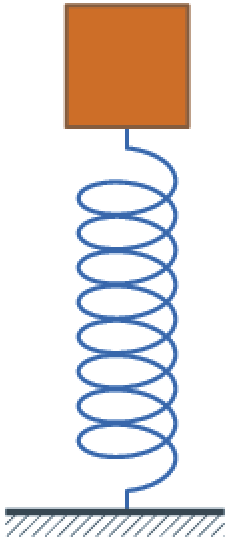
Static circuit
- ❑ COND1 - no storage element
- ❑ algebraic equations
- ❑ *DC circuit*

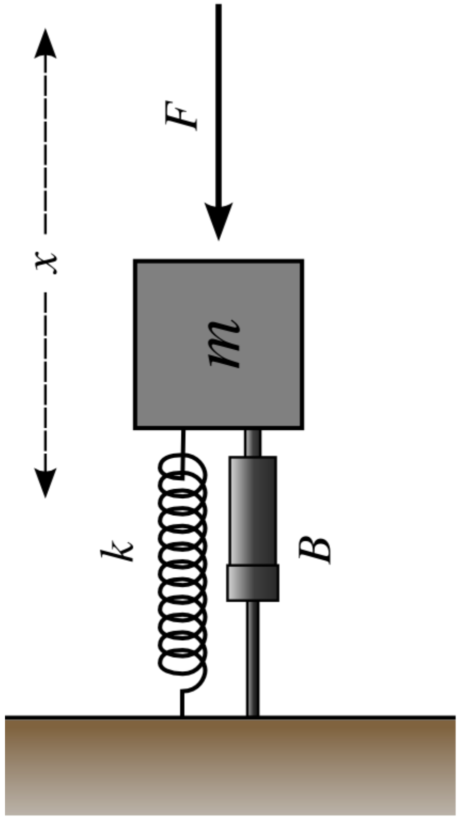
Dynamic circuit
- ❑ COND1: capacitor and/or inductor
- ❑ COND2: time var. source and/or time var. Structure *(switch)*
- ❑ differential equations
- ❑ *AC → part of DYN*

Mech. Component	**Electric Component**
Mass (m)	Inductor (L)
Spring (k)	Capacitor (C)
Shock absorber (B)	Resistor (R)
Force (F)	Voltage (V)
Velocity (dx/dt)	Current (dq/dt)

Transient Analysis 2

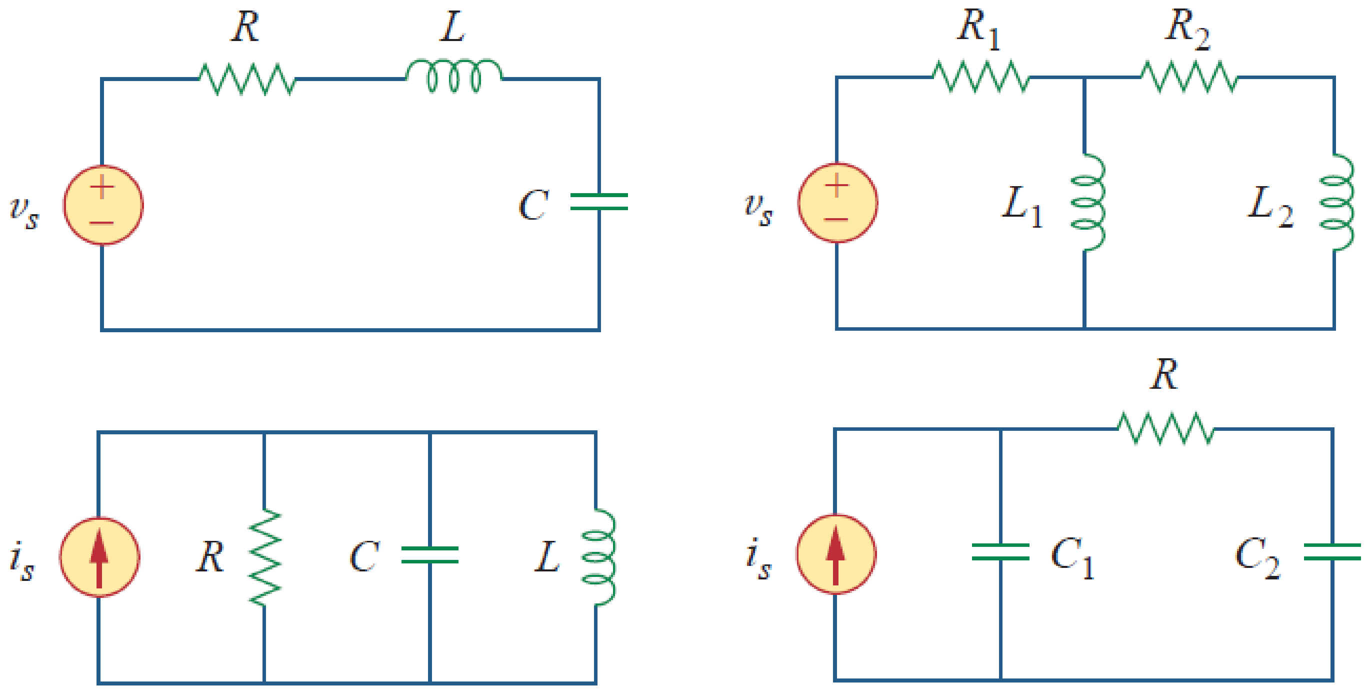
Second-order circuit
- ❑ *Two indeptendent (equivalent) reactive components*
- ❑ *Characterized by 2nd-order diff. equations*





MECHANICAL – ELECTRICAL ANALOGY

Introduction – Typical Second-Order Circuits

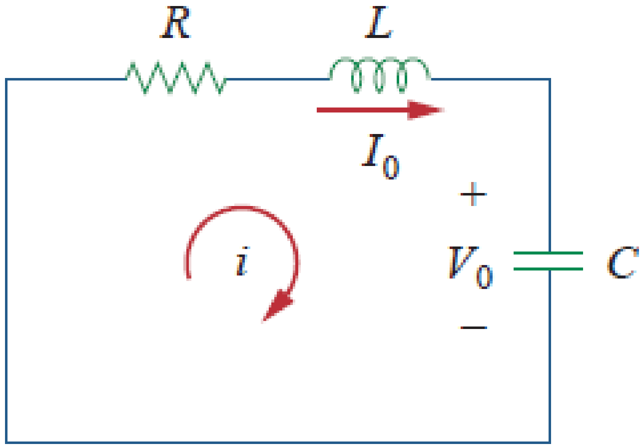


Progress in Content

- **The Source-Free Series RLC Circuit**
- Duality in Electric Circuits
- The Source-Free Parallel RLC Circuit
- Step Response of a Series RLC Circuit
- Step Response of a Parallel RLC Circuit
- General Second-Order Circuits

The Source-Free Series RLC Circuit

- ❑ Two separate ‚legs' of initial energy → two initial conditions
- ❑ To solve second-order equation there must be two initial values *(known from maths)*



$$v_L = L\frac{di}{dt}, \qquad i = C\frac{dv_C}{dt}$$

$$v_L + v_R + v_C = 0 \qquad v_L + RC\frac{dv_C}{dt} + v_C = 0$$

$$v_L = LC\frac{d^2v_C}{dt^2} \qquad LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = 0$$

$$w_L(0) = \frac{1}{2}Li_L^2(0) \rightarrow i_L(0) = I_0$$

$$w_C(0) = \frac{1}{2}Cv_C^2(0) \rightarrow v_C(0) = V_0$$

$$\frac{dv_C(0)}{dt} = \frac{1}{C}i(0) = \frac{I_0}{C}$$

‚Complementary' solution

$$v_C(t) = Ke^{st}$$

- ❑ *K - from initial conditions*
- ❑ *s – from coefficients of diff. equation*

The Source-Free Series RLC Circuit

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = 0$$

- Complementary solution form $\quad v_C(t) = Ke^{st} \rightarrow Ke^{st}(LCs^2 + RCs + 1) = 0$
- Characteristic equation $\quad LCs^2 + RCs + 1 = 0$
- Quadratic characteristic eq. → two roots $\quad s_{1,2} = \dfrac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC} = -\dfrac{R}{2L} \pm \sqrt{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC}}$
- Introducing damping factor and natural frequency $\quad \delta = \dfrac{R}{2L}, \qquad \omega_0 = \dfrac{1}{\sqrt{LC}} \rightarrow s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$
- Each root contributes in complementary solution $\quad v_C(t) = K_1e^{s_1t} + K_2e^{s_2t}$

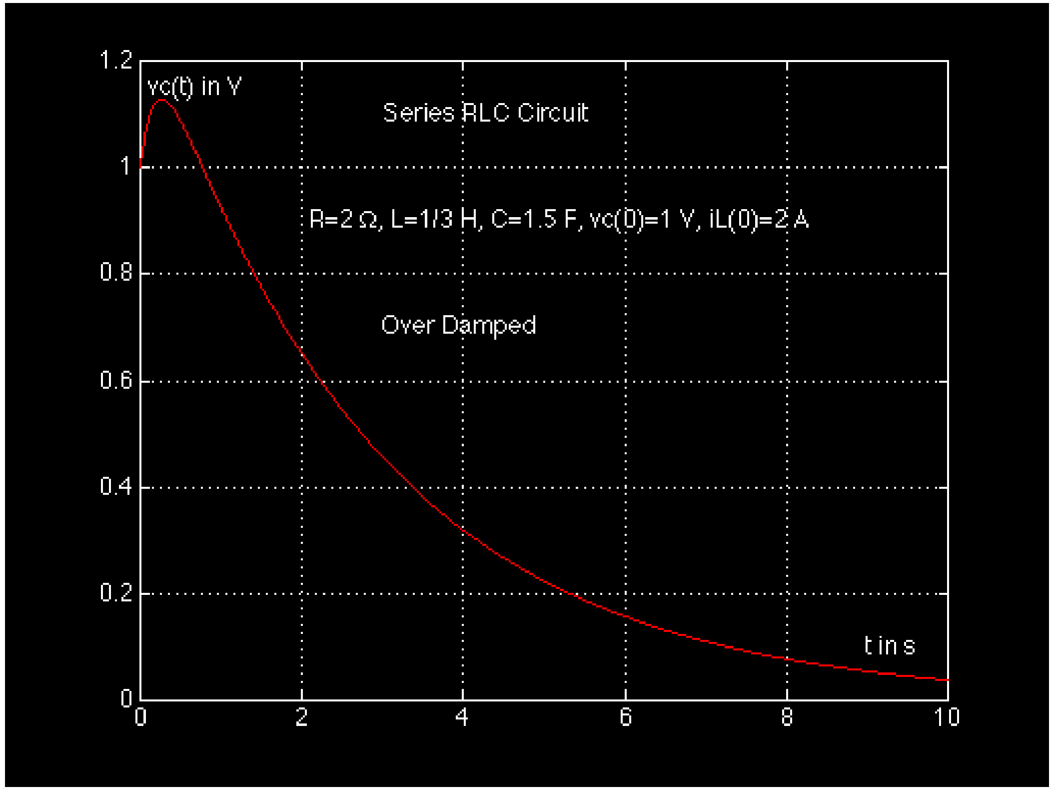
The Source-Free Series RLC Circuit

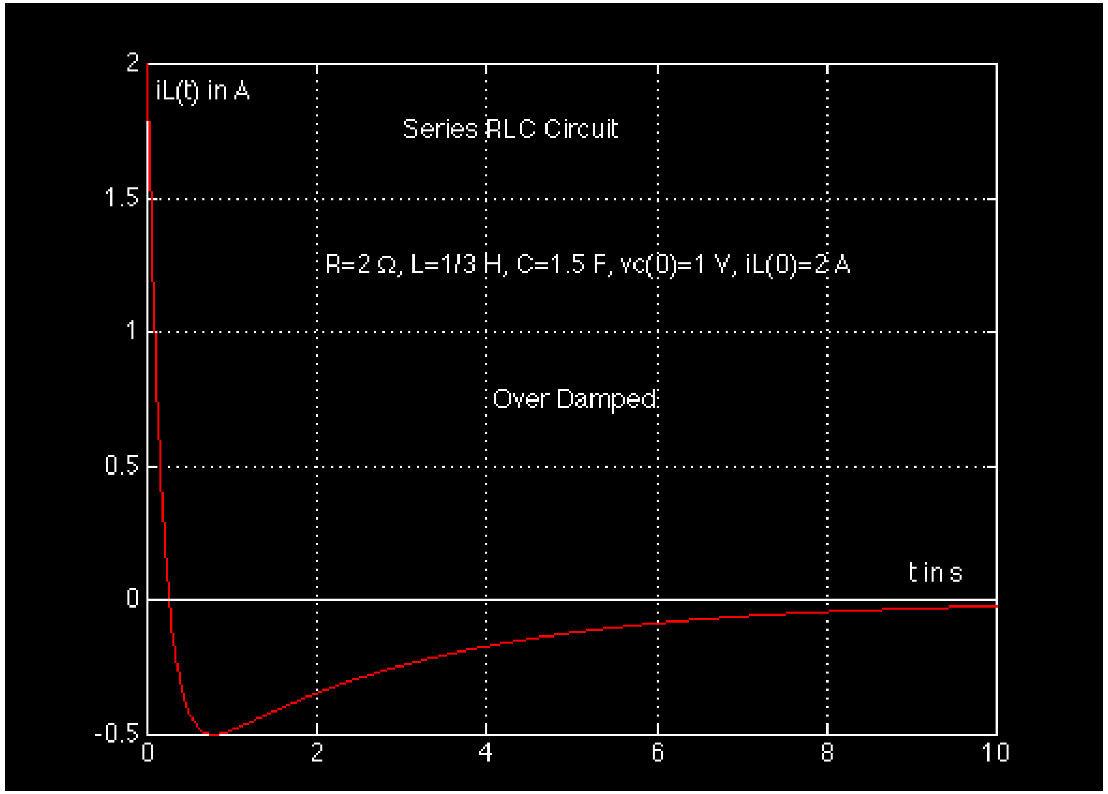
$$s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

CASE A	CASE B	CASE C	CASE D
$\delta^2 - \omega_0^2 > 0$	$\delta^2 - \omega_0^2 = 0$	$\delta^2 - \omega_0^2 < 0$	$\delta = 0$
$s_1 = -\delta + \sqrt{\delta^2 - \omega_0^2}$ $s_2 = -\delta - \sqrt{\delta^2 - \omega_0^2}$	$s_1 = s_2 = -\delta$	$s_1 = -\delta - j\sqrt{\omega_0^2 - \delta^2}$ $s_2 = -\delta + j\sqrt{\omega_0^2 - \delta^2}$	$s_1 = +j\omega_0$ $s_2 = -j\omega_0$
Two real unequal roots Overdamped circuit	Two real equal roots Critically damped circuit	Two cpx. conjugate roots Underdamped circuit	Two imaginary roots Undamped circuit (free-oscillation)

Case A – Overdamped Circuit

$$\delta^2 - \omega_0^2 > 0 \quad \rightarrow \quad s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} \quad \rightarrow \quad v_C(t) = K_1 e^{\left(-\delta + \sqrt{\delta^2 - \omega_0^2}\right)t} + K_2 e^{\left(-\delta - \sqrt{\delta^2 - \omega_0^2}\right)t}$$





$$v_C(t) = (1.318e^{-0.35t} - 0.318e^{-5.646t})\ V, \qquad i_L(t) = (-0.7e^{-0.35t} + 2.7e^{-5.646t})\ A$$

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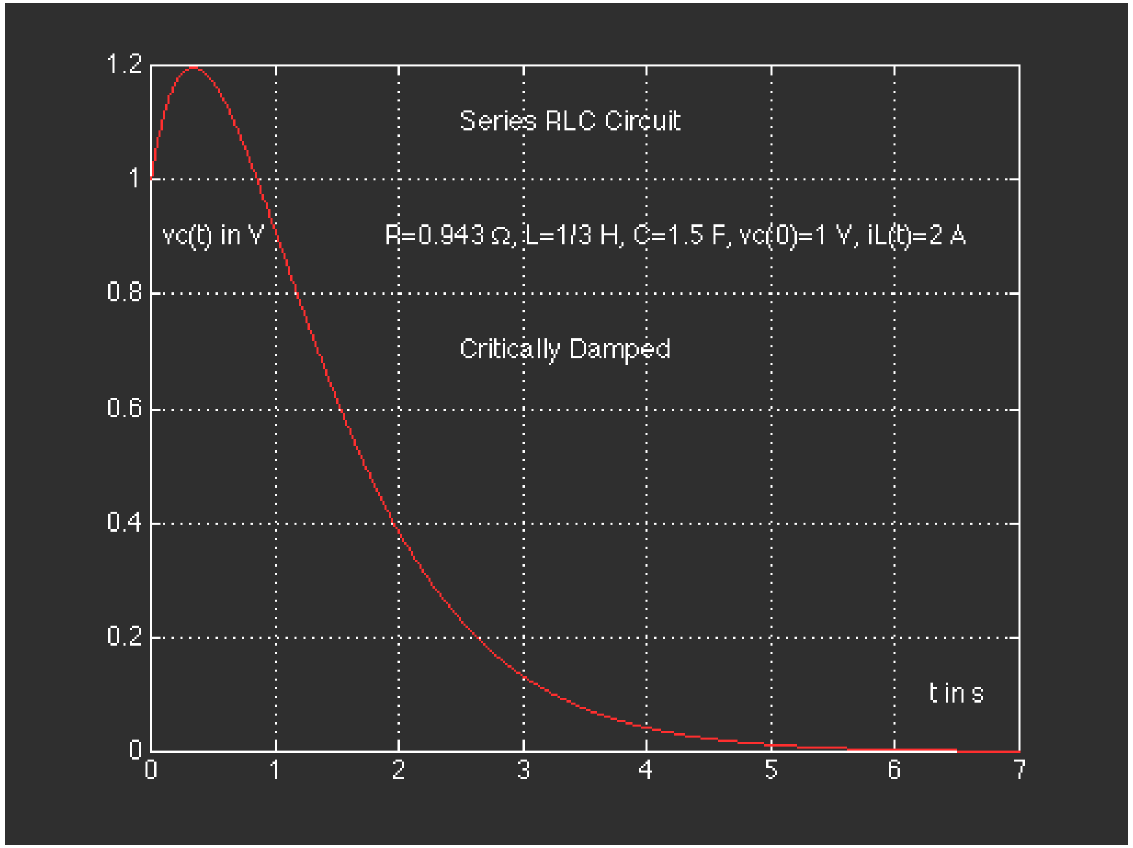
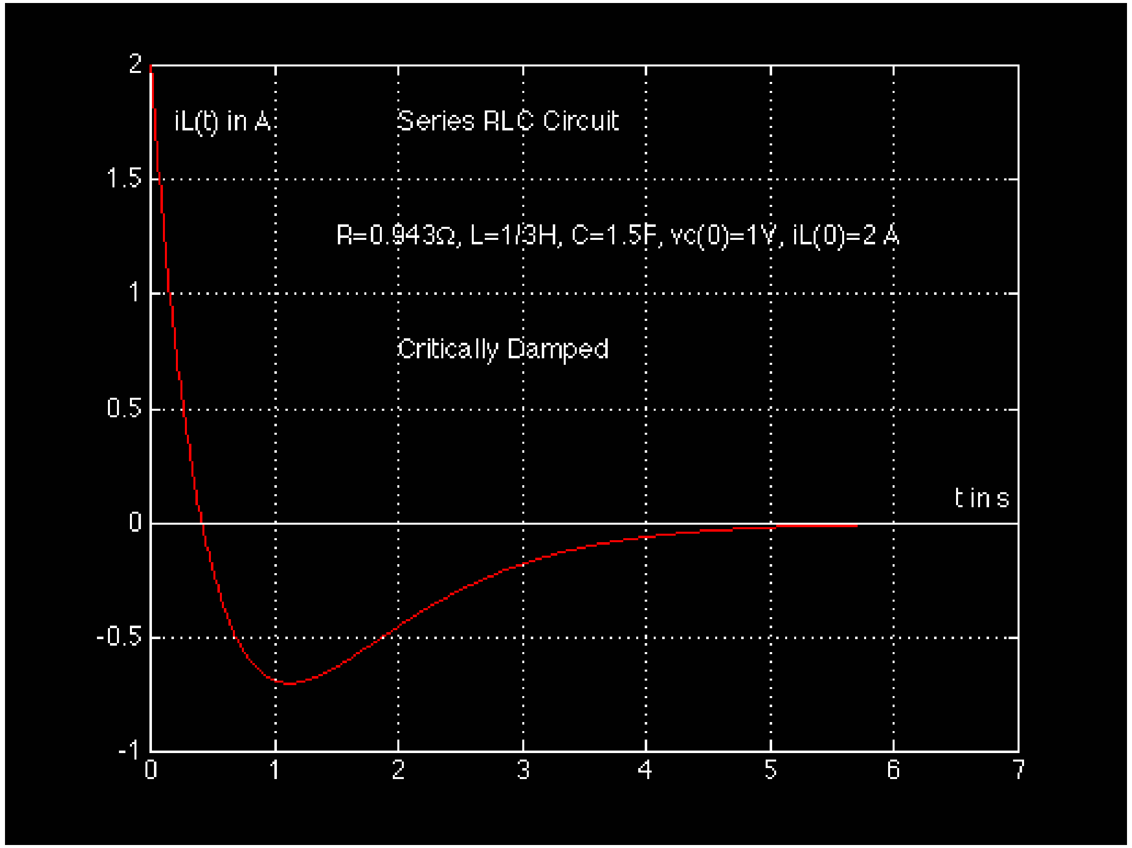
Case B – Critically Damped Circuit

$$\delta^2 - \omega_0^2 = 0 \quad \rightarrow \quad s_1 = s_2 = -\delta \quad \rightarrow \quad v_C(t) = K_1 e^{-\delta t} + K_2 t e^{-\delta t}$$

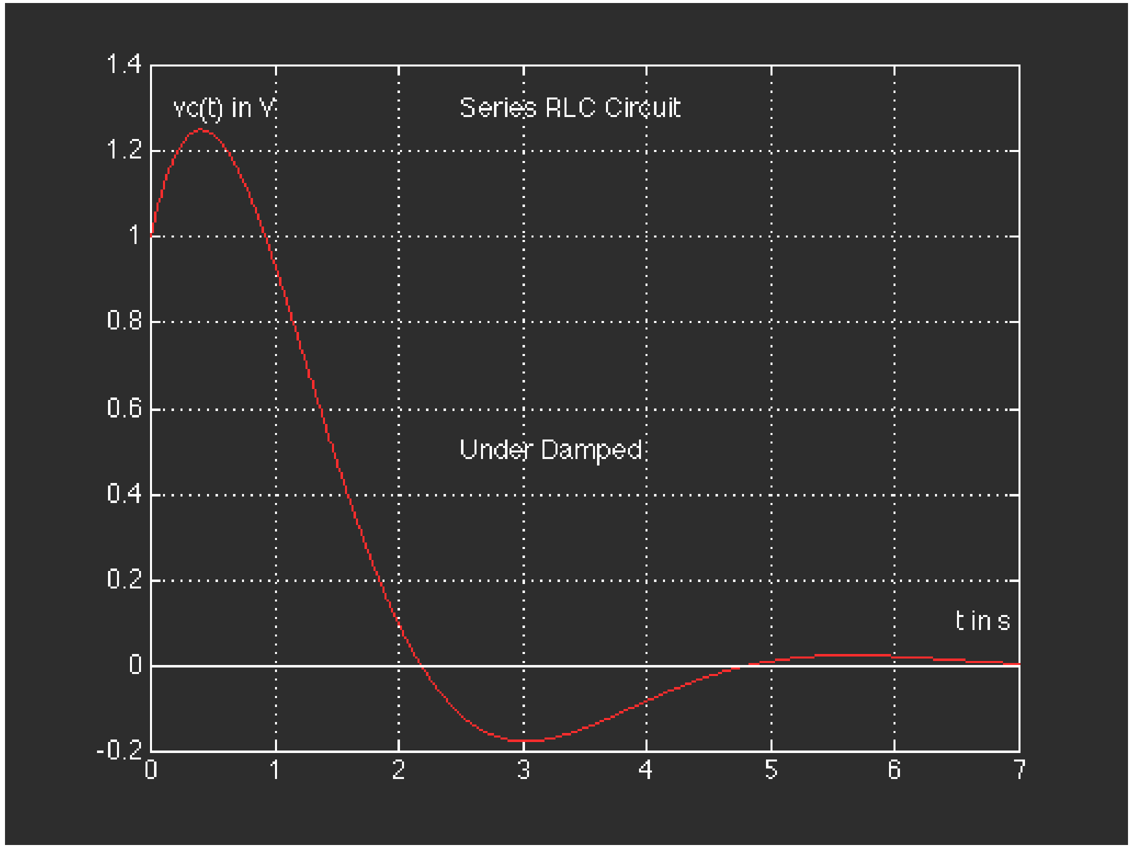


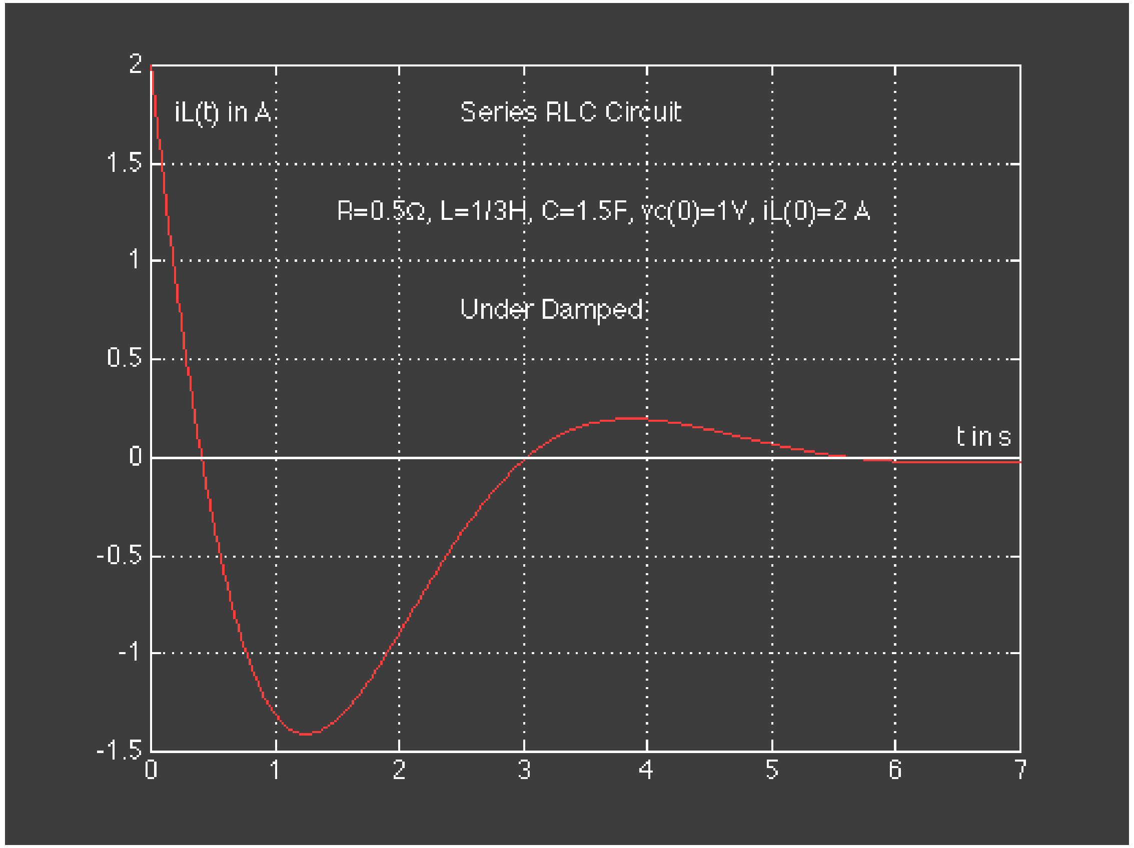


$$v_C(t) = (1.0e^{-1.414t} + 2.75te^{-1.414t})\ V, \qquad i_L(t) = (2e^{-1.414t} - 5.83te^{-1.414t})\ A$$

Case C – Underdamped Circuit

$$\delta^2 - \omega_0^2 < 0 \rightarrow s_{1,2} = -\delta \pm j\sqrt{\omega_0^2 - \delta^2} = -\delta \pm j\omega_d \rightarrow v_C(t) = e^{-\delta t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) = Ae^{-\delta t} \cos(\omega_d t + \varphi)$$

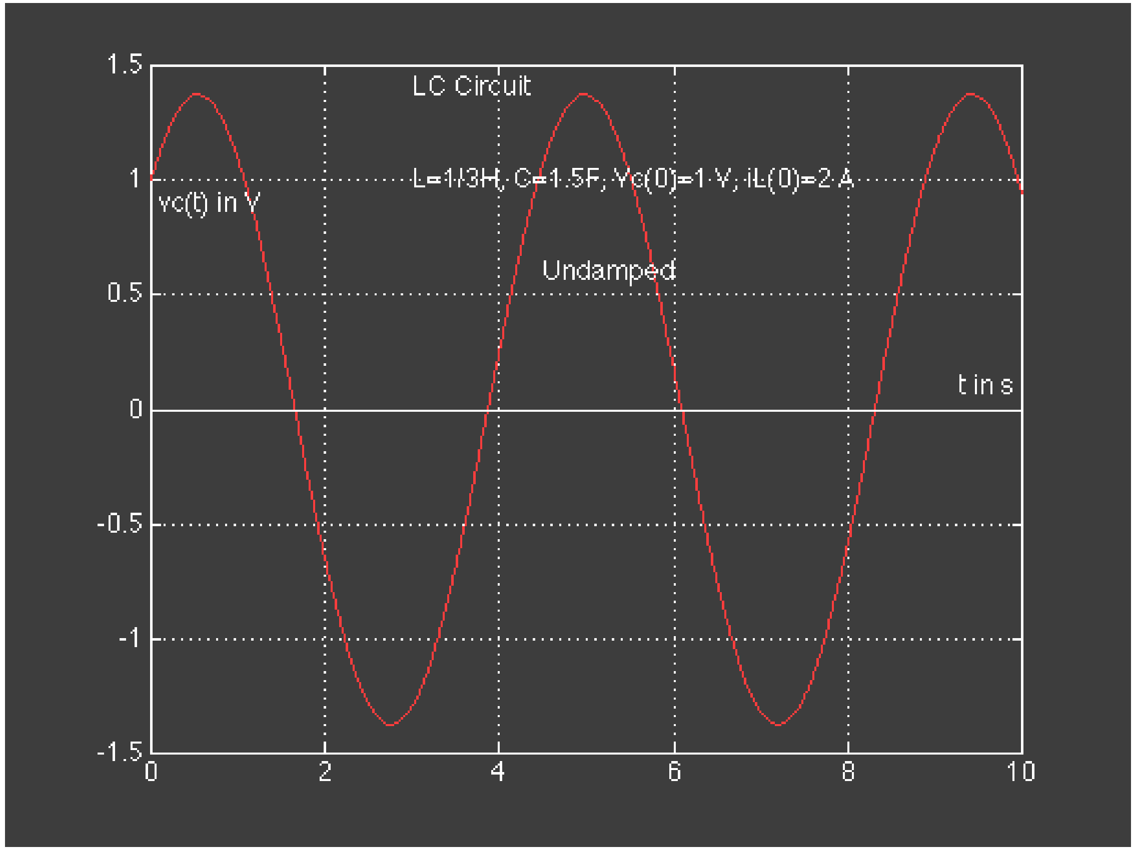


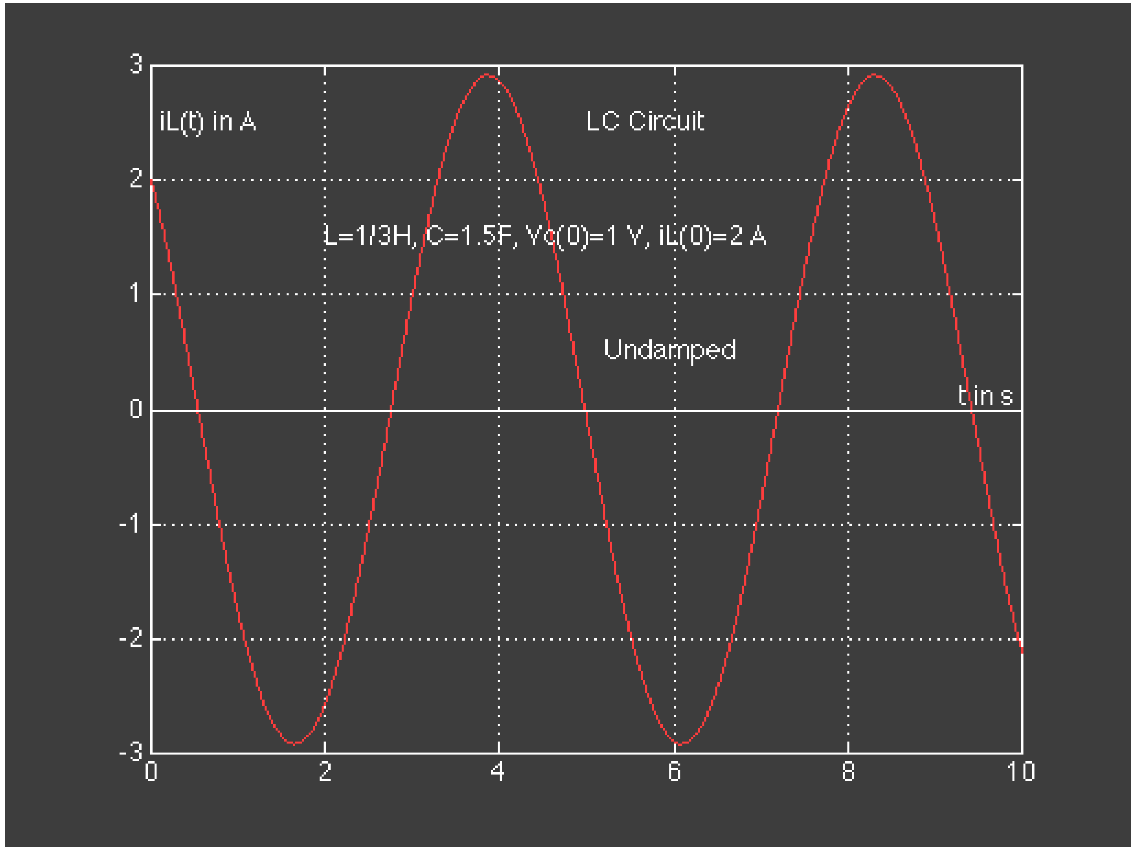


$$v_C(t) = 2e^{-0.75t} \cos(1.2t - 1.047)\ V, \qquad i_L(t) = 4.25e^{-0.75t} \cos(1.2t + 1.081)\, A$$

Case D – Undamped Circuit

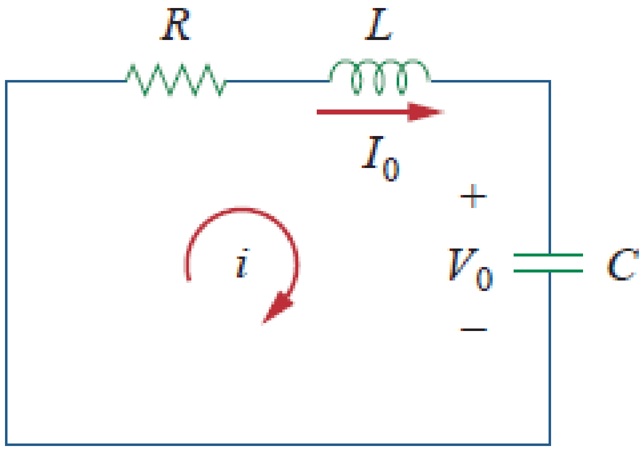
$$\delta = 0 \rightarrow s_{1,2} = \pm j\omega_0 \rightarrow v_C(t) = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t = A \cos(\omega_0 t + \varphi)$$

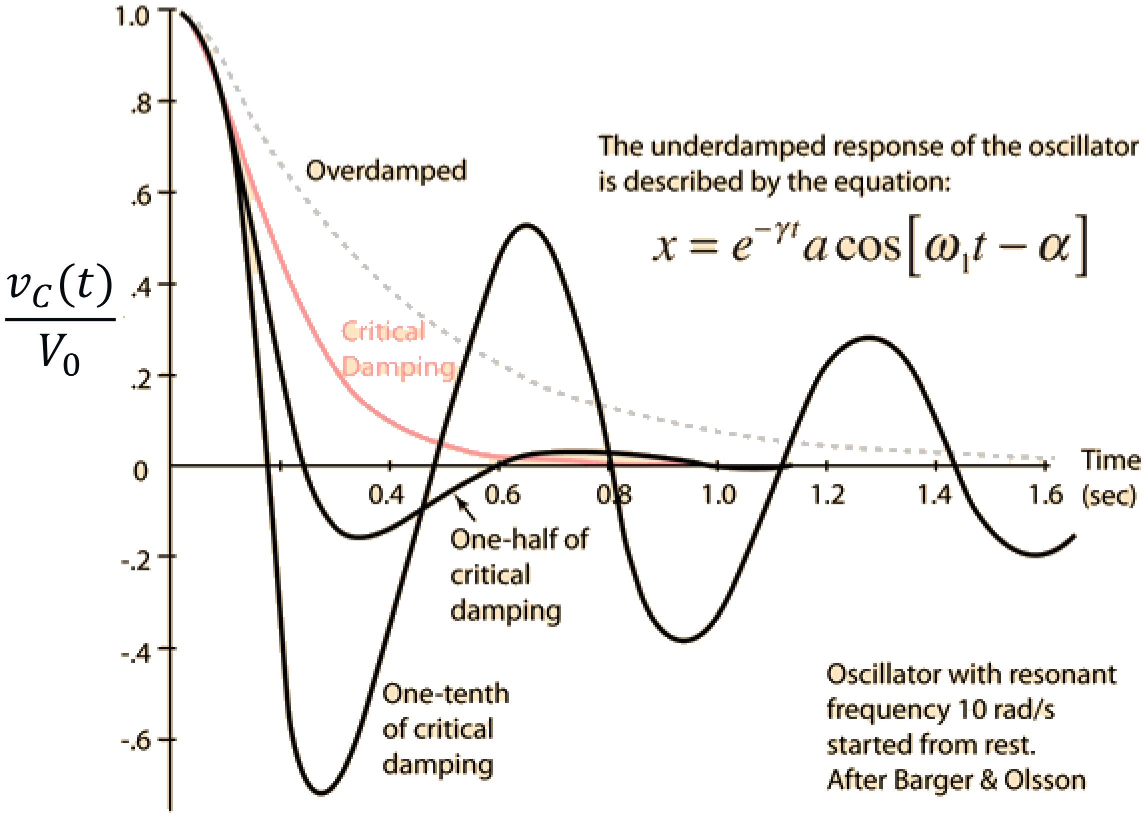




$$v_C(t) = 1.374 \cos(1.414t - 0.756)\ V, \qquad i_L(t) = 2.915 \cos(1.414t + 0.815)\, A$$

The Source-Free Series RLC Circuit (Summary)





[undamped] **vs.** [damped]

natural frequency; $\omega_0 \geq \omega_d$

$$\omega_d = \sqrt{\omega_0^2 - \delta^2}$$

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Electrical Duality

Duality in circuit theory (A. Russell, 1904).

- ❑ Parallelism bw. pairs of characterizing equations and theorems of circuits
- ❑ Interchanging dual parameters in an expression
- ❑ Dual expression has the same form

Examples

- ❑ *Duality of electricity and magnetism*
- ❑ $\sum v_i = 0 \quad \leftrightarrow \quad \sum i_i = 0$
- ❑ $\sum R_i = R_{eq} \quad \leftrightarrow \quad \sum G_i = G_{eq}$
- ❑ $v_n = v_0 \frac{R_n}{\sum R_i} \quad \leftrightarrow \quad i_n = i_0 \frac{G_n}{\sum G_i}$
- ❑ $v_L = L\frac{di_L}{dt} \quad \leftrightarrow \quad i_C = C\frac{dv_C}{dt}$

Dual Pair 1	**Dual Pair 2**
Resistance	Conductance
Inductance	Capacitance
Voltage	Current
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton

Elecrical Duality

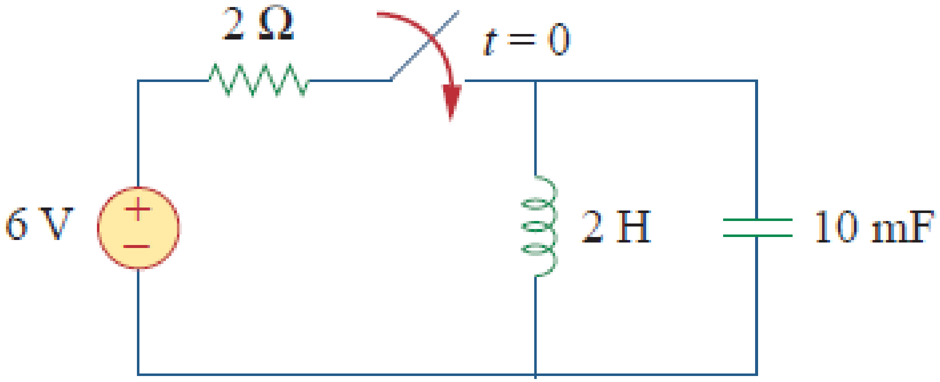
Dual circuits → same characterizing equations with interchanged dual quantities

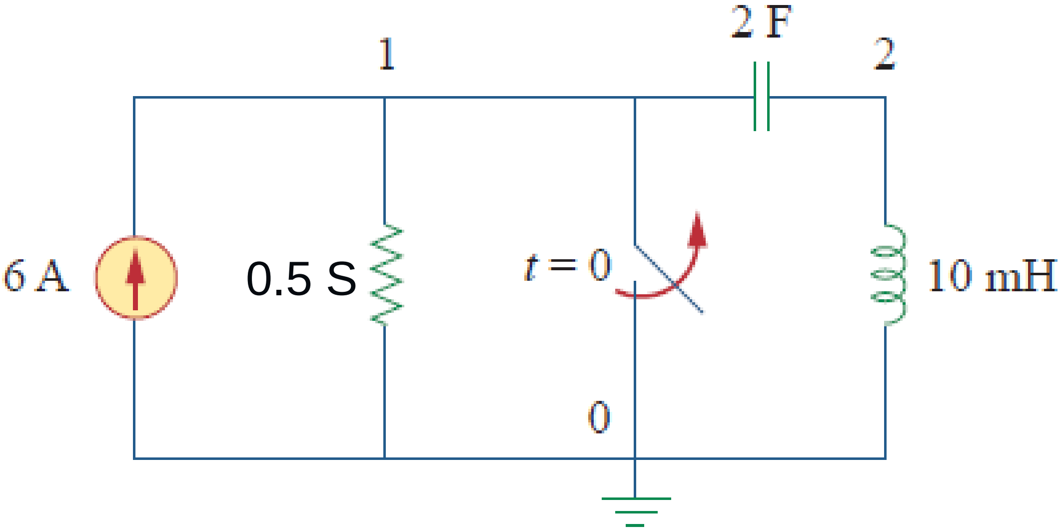
Notice - mutual inductance (for example) has no dual

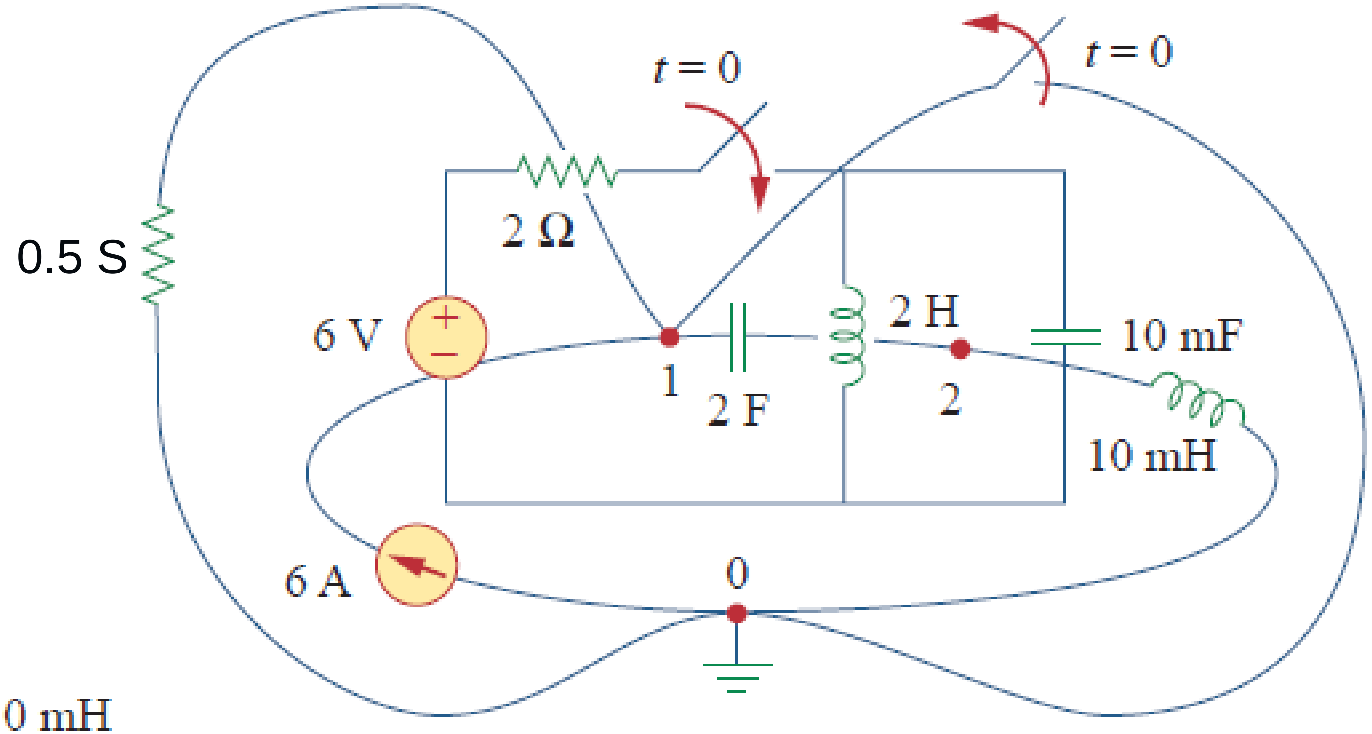
Graphical technique to find the dual circuit

- ❑ Place a node at the center of each mesh of the given circuit.
  Place the reference node (the ground) of the dual circuit outside the given circuit.
- ❑ Draw lines between the nodes such that each line crosses an element.
  Replace that element by its dual.
- ❑ To determine the polarity of voltage sources and direction of current sources, follow this rule: A voltage source that produces a positive (clockwise) mesh current has as its dual a current source whose reference direction is from the ground to the nonreference node.

Find the dual circuit.

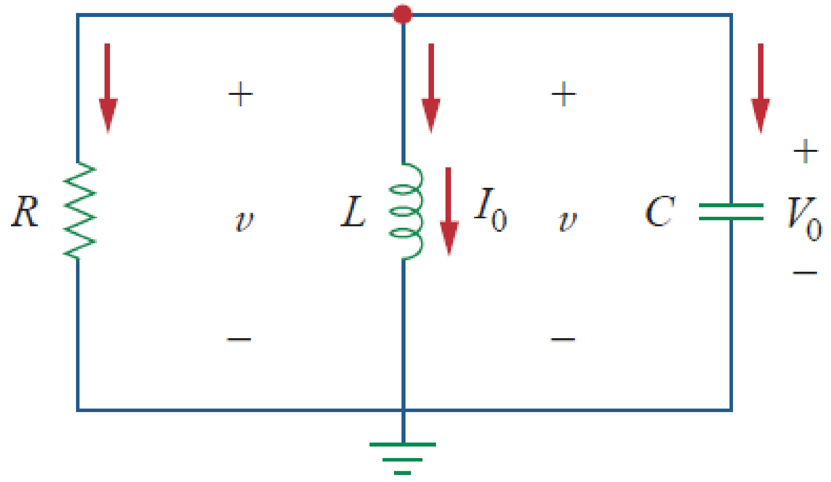






Progress in Content

The Source-Free Parallel RLC Circuit



$$v = L\frac{di_L}{dt}, \qquad i_C = C\frac{dv}{dt}$$

$$i_L + i_R + i_C = 0 \qquad i_L + GL\frac{di_L}{dt} + i_C = 0$$

$$i_C = LC\frac{d^2 i_L}{dt^2} \qquad LC\frac{d^2 i_L}{dt^2} + GL\frac{di_L}{dt} + i_L = 0$$

Initial conditions
To solve second-order equation
there must be two initial values.

$$i_L(0) = I_0$$

$$\frac{di_L(0)}{dt} = \frac{1}{L}v(0) = \frac{V_0}{L}$$

‚Complementary' solution

$$i_L(t) = Ke^{st}$$

- ❑ *K - from initial conditions*
- ❑ *s – from coefficients of diff. equation*

The Source-Free Parallel RLC Circuit

$$LC\frac{d^2 i_L}{dt^2} + GL\frac{di_L}{dt} + i_L = 0$$

- ❑ Complementary solution form $\quad i_L(t) = Ke^{st} \rightarrow Ke^{st}(LCs^2 + GLs + 1) = 0$
- ❑ Characteristic equation $\quad LCs^2 + GLs + 1 = 0$
- ❑ Quadratic characteristic eq. → two roots $\quad s_{1,2} = \dfrac{-GL \pm \sqrt{(GL)^2 - 4LC}}{2LC} = -\dfrac{G}{2C} \pm \sqrt{\left(\dfrac{G}{2C}\right)^2 - \dfrac{1}{LC}}$
- ❑ Introducing damping factor and natural frequency $\quad \delta = \dfrac{G}{2C}, \qquad \omega_0 = \dfrac{1}{\sqrt{LC}} \rightarrow s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$
- ❑ Each root contributes in complementary solution $\quad i_L(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

The Source-Free Parallel RLC Circuit

$$s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

CASE A	CASE B	CASE C	CASE D
$\delta^2 - \omega_0^2 > 0$	$\delta^2 - \omega_0^2 = 0$	$\delta^2 - \omega_0^2 < 0$	$\delta = 0$
$s_1 = -\delta + \sqrt{\delta^2 - \omega_0^2}$ $s_2 = -\delta - \sqrt{\delta^2 - \omega_0^2}$	$s_1 = s_2 = -\delta$	$s_1 = -\delta - j\sqrt{\omega_0^2 - \delta^2}$ $s_2 = -\delta + j\sqrt{\omega_0^2 - \delta^2}$	$s_1 = +j\omega_0$ $s_2 = -j\omega_0$
Two real unequal roots Overdamped circuit	Two real equal roots Critically damped circuit	Two cpx. conjugate roots Underdamped circuit	Two imaginary roots Undamped circuit (free-oscillation)

Case A (overdamped)

$$\delta^2 - \omega_0^2 > 0 \quad \rightarrow \quad s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} \quad \rightarrow \quad i_L(t) = K_1 e^{\left(-\delta + \sqrt{\delta^2 - \omega_0^2}\right)t} + K_2 e^{\left(-\delta - \sqrt{\delta^2 - \omega_0^2}\right)t}$$

Case B (critically damped)

$$\delta^2 - \omega_0^2 = 0 \quad \rightarrow \quad s_1 = s_2 = -\delta \quad \rightarrow \quad i_L(t) = K_1 e^{-\delta t} + K_2 t e^{-\delta t}\,, \qquad t \geq 0$$

Case C (underdamped)

$$\delta^2 - \omega_0^2 < 0 \rightarrow s_{1,2} = -\delta \pm j\sqrt{\omega_0^2 - \delta^2} = -\delta \pm j\omega_d \rightarrow i_L(t) = e^{-\delta t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

Case D (undamped)

$$\delta = 0 \rightarrow s_{1,2} = \pm j\omega_0 \rightarrow i_L(t) = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t$$

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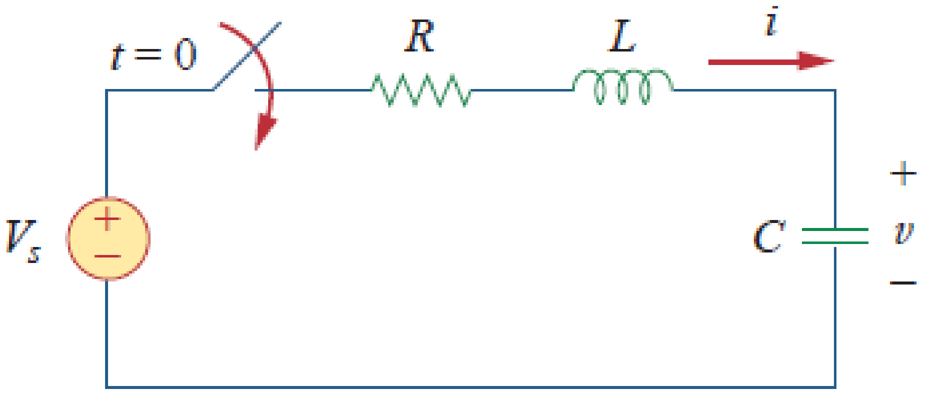
Step Response of a Series RLC Circuit



$$L\frac{di}{dt} + Ri + v = V_S, \qquad i = C\frac{dv}{dt}$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_S}{LC}$$

Complete response = transient response (*'temporary part, dies out w. time'*) + **steady-state response** (*‚permanent part'*)

$v(t) = v_{tr}(t) + v_{ss}(t)$

Transient part

- Overdamped $v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$
- Critically damped $v(t) = (K_1 + K_2)e^{-\delta t}$
- Underdamped $v(t) = (K_1 \cos \omega_d t + K_2 \sin \omega_d t)e^{-\delta t}$

Steady-state part $v_{ss}(t) = v(\infty) = V_S$

In general (for any x variable)

$$x(t) = x_{tr}(t) + x_{ss}(t)$$

$$K_1, K_2 \leftarrow x(0), \frac{dx(0)}{dt} \; initial\ conditions$$

$$s_1, s_2 \ldots (\delta, \omega_d) \leftarrow RLC\ circuit\ elements$$

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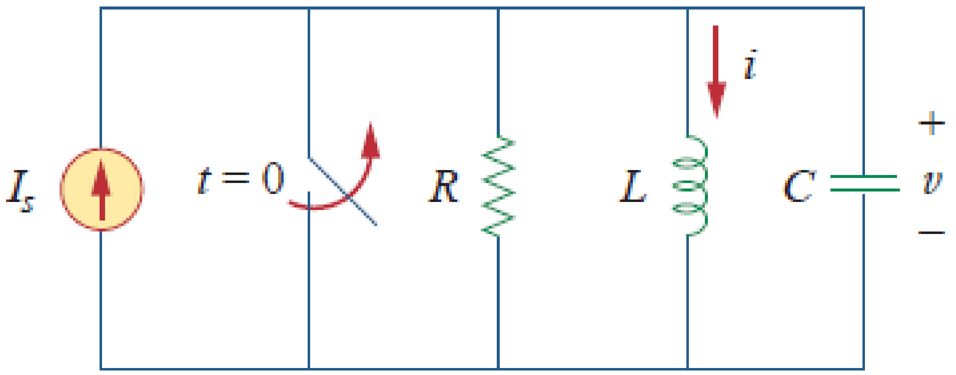
Step Response of a Parallel RLC Circuit



$$C\frac{dv}{dt} + i + \frac{v}{R} = I_S, \qquad v = L\frac{di}{dt}$$

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_S}{LC}$$

Complete response = steady-state response (*‚permanent part'*) + **transient response** (*'temporary part, dies out w. time'*)

$$i(t) = i_{ss}(t) + i_{tr}(t)$$

Complete response

- Overdamped $\quad i(t) = I_S + K_1 e^{s_1 t} + K_2 e^{s_2 t}$
- Critically damped $\quad i(t) = I_S + (K_1 + K_2)e^{-\delta t}$
- Underdamped $\quad i(t) = I_S + (K_1 \cos\omega_d t + K_2 \sin\omega_d t)e^{-\delta t}$

In general (for any x variable)

$$x(t) = x_{tr}(t) + x_{ss}(t)$$

$$K_1, K_2 \leftarrow x(0), \frac{dx(0)}{dt} \ \ initial\ conditions$$

$$s_1, s_2 \ldots (\delta, \omega_d) \leftarrow RLC\ circuit\ elements$$

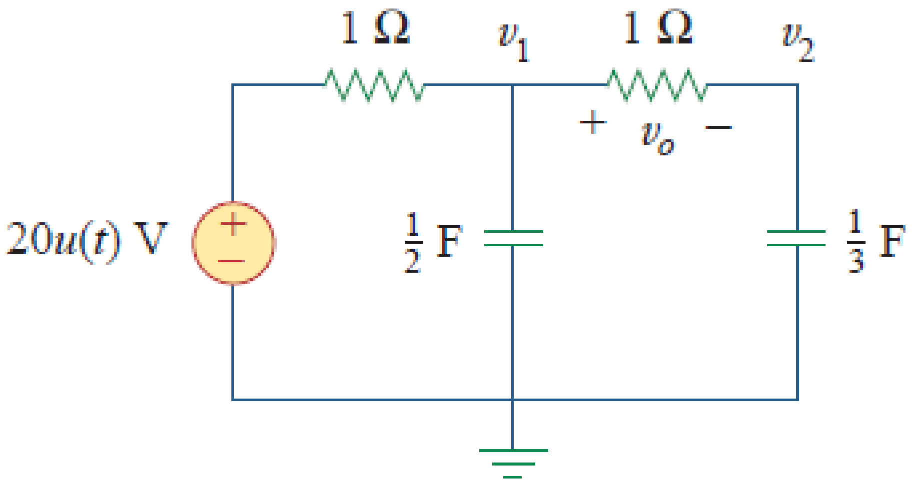
Progress in Content

- Find the initial conditions $x(0)$, $dx(0)/dt$ and the final value $x(\infty)$
- Turn off the independent sources. Find $x_{tr}(t)$ by KCL/KVL (*2nd-order diff. eq.)*
  → *characteristic roots (over-, critically, or underdamped)* → $x_{tr}(t)$ *w. 2 unknown constants*
- Obtain the steady-state response $x_{ss}(t) = x(\infty)$
- Total response $x(t) = x_{tr}(t) + x_{ss}(t)$
- Determine the constants in transient response by $x(0)$, $dx(0)/dt$.

Check your skill.

$v_0(t) =?$ for $t > 0$.
(Hint: first find v_1 and v_2.)

Result $8(e^{-t} - e^{-6t})\ V$



Questions