DR. GYURCSEK ISTVÁN

Overview of the Laplace Transform

Sources and additional materials (recommended)

- *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- *http://mathworld.wolfram.com/IntegralTransform.html*
- *https://en.wikipedia.org/wiki/Laplace_transform*
- *https://en.wikipedia.org/wiki/Fourier_transform*
- *https://en.wikipedia.org/wiki/Z-transform*

Progress in Content

- **Introduction to the Integral Transforms**
- The Laplace Transform
- Circuit Analysis using Laplace Transform
- Transfer Function in s-Domain
- The Convolution Integral
- Appendix: Table of Laplace Transforms

gyurcsek.istvan@mik.pte.hu

- ❑ Differential equations
  - Easy to understand
    → difficult to solve
    (even first order networks)
  - Initial conditions
    → can be difficult to calculate
- ❑ More convenient method
  → **INTEGRAL TRANSFORM**
  (theoretical approach)

$$F(s) = \int_{t_1}^{t_2} f(t) \cdot K(s,t) \cdot dt$$

K(s,t) - integral kernel of the transform

SEE ALSO:
Buschman Transform, **Fourier Transform**
Fourier-Stieltjes Transform, G-Transform
H-Transform, Hadamard Transform
Hankel Transform, Hartley Transform
Hough Transform
Kontorovich-Lebedev Transform
Laplace Transform
Mehler-Fock Transform
Meijer Transform,
Mellin Transform, Narain G-Transform
Operational Mathematics, Radon Transform
Stieltjes Transform, W-Transform, Wavelet Transform
Z-Transform

Progress in Content

- ❑ Introduction to the Integral Transforms
- ❑ **The Laplace Transform**
- ❑ Circuit Analysis using Laplace Transform
- ❑ Transfer Function in s-Domain
- ❑ The Convolution Integral
- ❑ Appendix: Table of Laplace Transforms

Laplace Transform

- Bilateral (two-sided) Laplace transform

$$\mathcal{L}[f(t)] = F(s) = \int_{-\infty}^{\infty} f(t) \cdot e^{-st} \cdot dt \ , \qquad s = \sigma + j\omega$$

- Unilateral (one-sided) Laplace transform *(practical life)*

$$\mathcal{L}[f(t)] = F(s) = \int_{0}^{\infty} f(t) \cdot e^{-st} \cdot dt \ , \qquad s = \sigma + j\omega$$

$$f(t) \rightarrow u(t)f(t) \ , \qquad f(t) \ ignored \ for \ t < 0$$

Convergence Criterion

$$\int_0^\infty |f(t)| \cdot |e^{-st}| \cdot dt < \infty \,, \qquad s = \sigma + j\omega$$

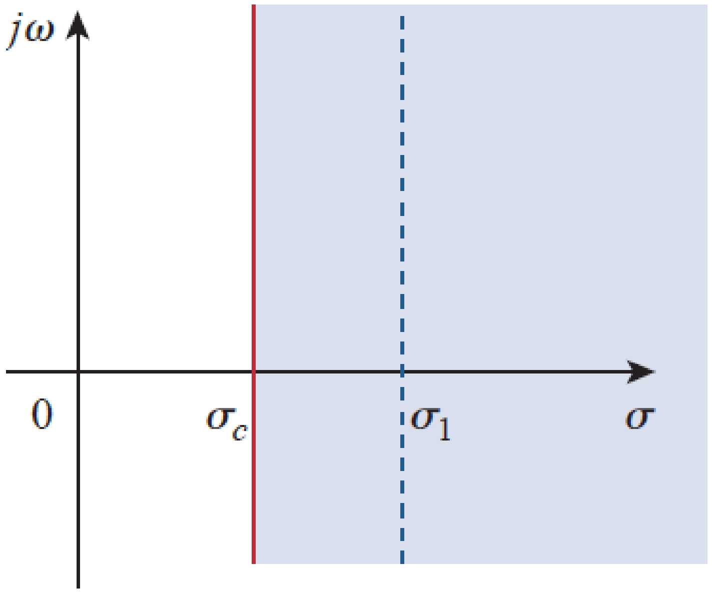
$$\left|e^{-j\omega t}\right| = \sqrt{cos^2\omega t + sin^2\omega t} = 1$$

$$\int_0^\infty e^{-\sigma t} \cdot |f(t)| \cdot dt < \infty$$

$$ROC: Re\{s\} = \sigma > \sigma_c \rightarrow F(s)\ exists$$

$$meaning\ ({}^{w}/_{o}\ prove) \rightarrow |f(t)| \leq K \cdot e^{\sigma_c t}$$

ROC (Region of Convergence)



All functions of interest in circuit analysis satisfy the convergence criterion.

gyurcsek.istvan@mik.pte.hu

Inverse Laplace Transform

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}ds$$

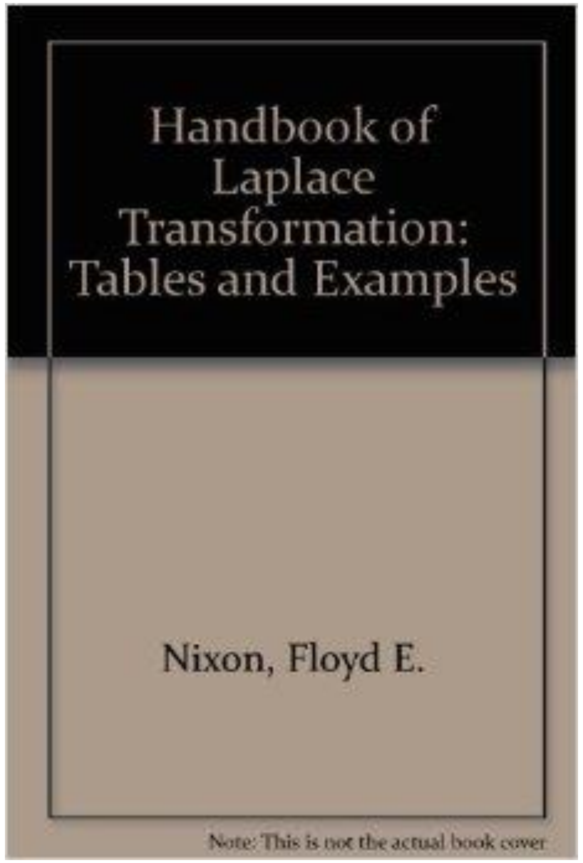
- integration is performed along a straight line in ROC

$$\sigma_1 > \sigma_c\,, -\infty < \omega < \infty$$

F(s) ⇔ f(t)

In practice

LOOK-UP TABLE instead of CPX ANALYSIS



gyurcsek.istvan@mik.pte.hu

Properties of Laplace Transform *(Selection)*

Linearity $\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$

Scaling $\mathcal{L}[f(at)] = \frac{1}{a} F(\frac{s}{a})$

$$\mathcal{L}[f(at)] = \int_0^\infty f(at) e^{-st} dt$$

$$\mathcal{L}[f(x)] = \int_0^\infty f(x) e^{-x\left(\frac{s}{a}\right)} \frac{dx}{a} \leftarrow x = a \cdot t$$

$$= \frac{1}{a} \int_0^\infty f(x) e^{-x\left(\frac{s}{a}\right)} dx = \frac{1}{a} F(\frac{s}{a})$$

Time shift (w/o prove) $\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s)$

Frequency shift (w/o prove) $\mathcal{L}[e^{-at} f(t)u(t)] = F(s+a)$

Time differentiation $\mathcal{L}[f'(t)] = sF(s) - f(0)$

Time integration $\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s} F(s)$

Unit step (w/o prove) $\mathcal{L}\left[\int_0^t u(t)dt\right] = \frac{1}{s}$

http://en.wikipedia.org/wiki/Laplace_transform

Laplace Transform Application – Integrodifferential Equations

- Differentiation and integration property → useful to solve (integro)diff. eqs.
- Example → solve the diff. eq. when $v(0) = 1$ and $v'(t) = -2$

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t) \qquad [s^2V(s) - sv(0) - v'(0)] + 6[sV(s) - v(0)] + 8V(s) = \frac{2}{s}$$

$$v(0) = 1, v'(t) = -2 \rightarrow s^2V(s) - s + 2 + 6sV(s) - 6 + 8V(s) = \frac{2}{s}$$

$$\rightarrow (s^2 + 6s)V(s) = s + 4 + \frac{2}{s} = \frac{s^2 + 4s + 2}{s}$$

$$V(s) = \frac{s^2 + 4s + 2}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$A = sV(s)\Big|_{s=0} = \frac{s^2 + 4s + 2}{(s+2)(s+4)}\Bigg|_{s=0} = \frac{2}{2 \cdot 4} = \frac{1}{4}$$

⇧ Partial-faction decomposition

$$B = (s+2)V(s)\Big|_{s=-2} = \frac{s^2 + 4s + 2}{s(s+4)}\Bigg|_{s=-2} = \frac{-2}{2 \cdot (-2)} = \frac{1}{2}$$

$$V(s) = \frac{1/4}{s} + \frac{1/2}{s+2} + \frac{1/4}{s+4}$$

$$C = (s+4)V(s)\Big|_{s=-4} = \frac{s^2 + 4s + 2}{s(s+2)}\Bigg|_{s=-4} = \frac{2}{(-4) \cdot (-2)} = \frac{1}{4}$$

$$v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$

Progress in Content

- Introduction to the Integral Transforms
- The Laplace Transform
- **Circuit Analysis using Laplace Transform**
- Transfer Function in s-Domain
- The Convolution Integral
- Appendix: Table of Laplace Transforms

Circuit Element Models with Initial Conditions

(1) Resistor $v(t) = R \cdot i(t) \rightarrow \mathcal{L}[v(t)] = \mathcal{L}[R \cdot i(t)] \rightarrow V(s) = R \cdot I(s)$

(2) Inductor $v(t) = L \cdot \frac{di(t)}{dt} \rightarrow \mathcal{L}[v(t)] = \mathcal{L}\left[L \cdot \frac{di(t)}{dt}\right] \rightarrow V(s) = L \cdot [s \cdot I(s) - i(0^-)] = s \cdot L \cdot I(s) - L \cdot i(0^-)$

$$V(s) = s \cdot L \cdot I(s) - L \cdot i(0^-) \text{ or } I(s) = \frac{1}{sL} V(s) + \frac{i(0^-)}{s}$$

(3) Capacitor $i(t) = C \cdot \frac{dv(t)}{dt} \rightarrow \mathcal{L}[i(t)] = \mathcal{L}\left[C \cdot \frac{dv(t)}{dt}\right] \rightarrow I(s) = C \cdot [s \cdot V(s) - v(0^-)] = s \cdot C \cdot V(s) - C \cdot v(0^-)$

$$I(s) = s \cdot C \cdot V(s) - C \cdot v(0^-) \text{ or } V(s) = \frac{1}{sC} I(s) + \frac{v(0^-)}{s}$$

(4) Operator impedance
(under zero initial conditions!)

$$Z_R(s) = \frac{V(s)}{I(s)} = R\,, \qquad Z_L(s) = \frac{V(s)}{I(s)} = s \cdot L\,, \qquad Z_C(s) = \frac{V(s)}{I(s)} = \frac{1}{s \cdot C}$$

gyurcsek.istvan@mik.pte.hu

Circuit Element Models with Initial Conditions

Inductor

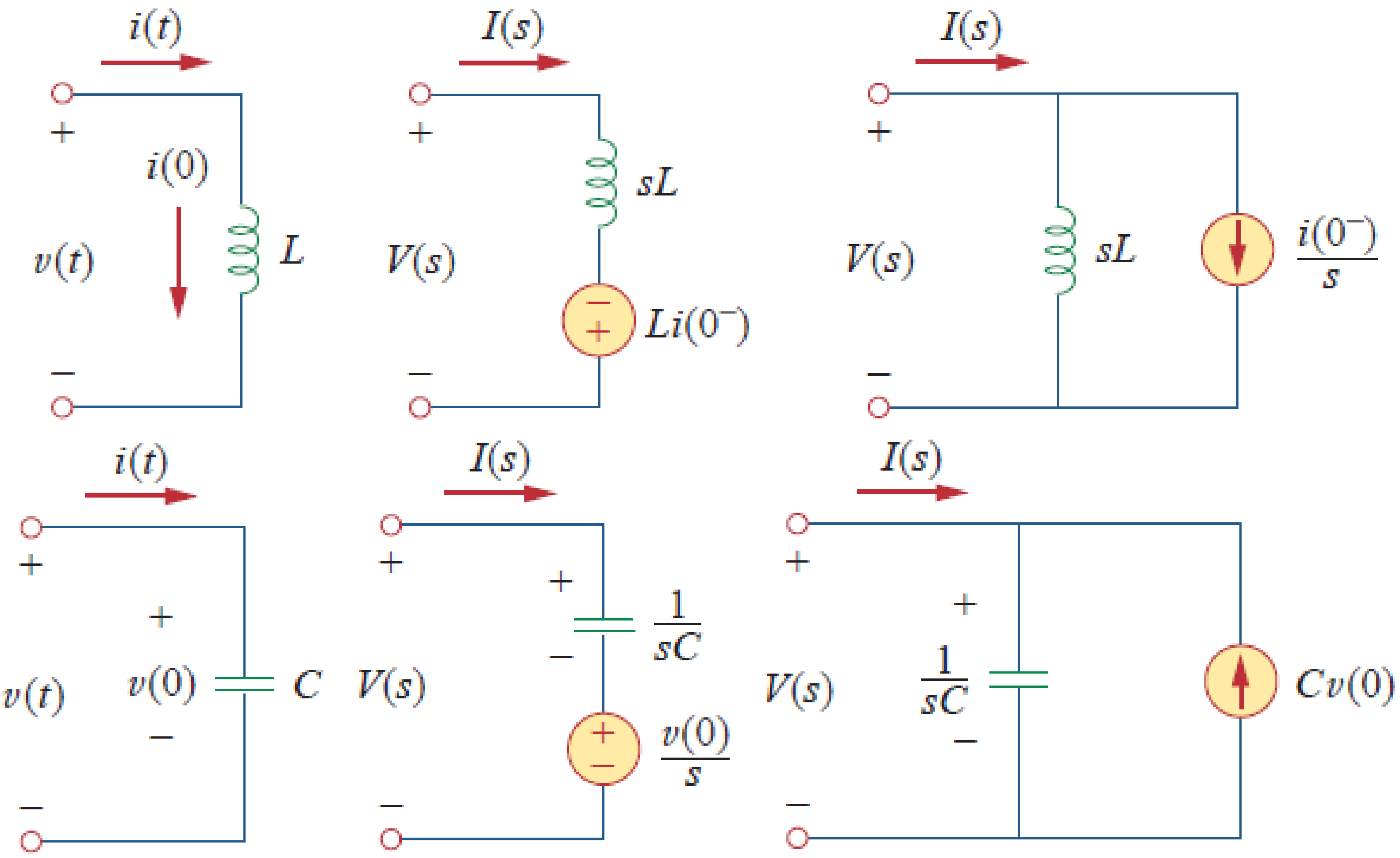
$$V(s) = s \cdot L \cdot I(s) - L \cdot i(0^-)$$

$$I(s) = \frac{1}{sL} V(s) + \frac{i(0^-)}{s}$$

Capacitor

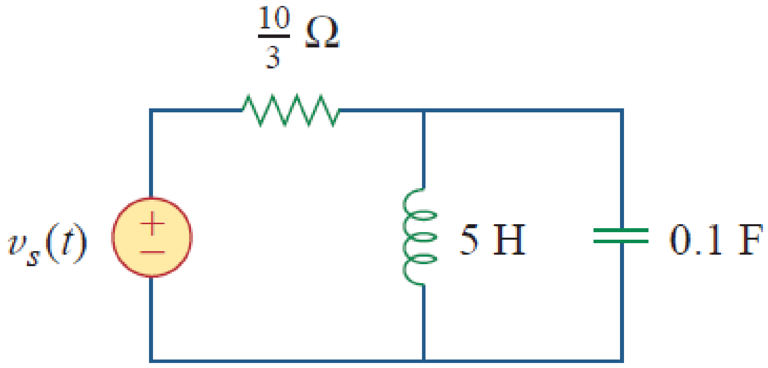
$$I(s) = s \cdot C \cdot V(s) - C \cdot v(0^-)$$

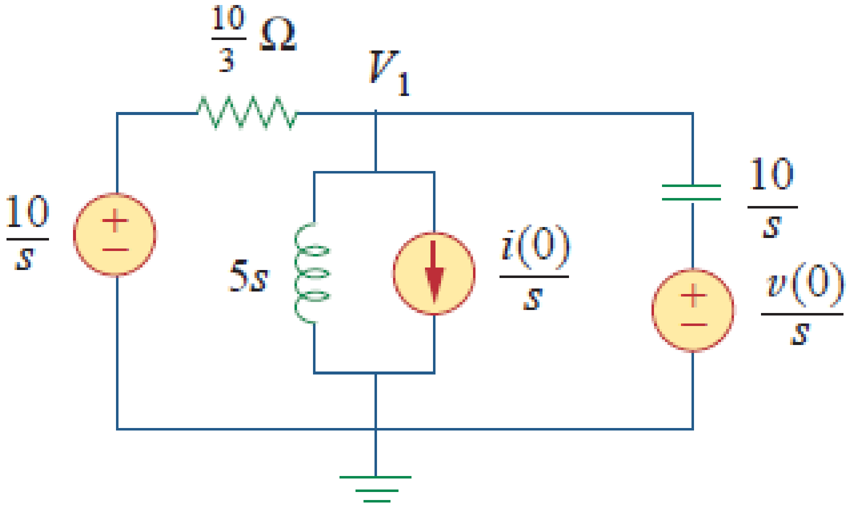
$$V(s) = \frac{1}{sC} I(s) + \frac{v(0^-)}{s}$$



gyurcsek.istvan@mik.pte.hu

Circuit Analysis - Example





$v_S(t) = 10u(t)\,V, i_L(0) = -1\,A, v_C(0) = 5\,V, v_C(t) = ?$

$$\frac{V_1 - {}^{10}\!/_{s}}{{}^{10}\!/_{3}} + \frac{V_1 - 0}{5s} + \frac{i(0)}{s} + \frac{V_1 - \left[{}^{v(0)}\!/_{s}\right]}{{}^{1}\!/_{0.1s}} = 0$$

$$V_1 = \frac{40 + 5s}{s^2 + 3s + 2} = \frac{40 + 5s}{(s+1)(s+2)} = \cdots = \frac{35}{s+1} - \frac{30}{s+2}$$

$$0.1\left(s + 3 + \frac{2}{s}\right)V_1 = \frac{3}{s} + \frac{1}{s} + 0.5 \rightarrow (s^2 + 3s + 2)V_1 = 40 + 5s$$

$$\mathcal{L}^{-1}[V_1(s)] = v_1(t) = (35e^{-t} - 30e^{-2t})u(t)\,V$$

gyurcsek.istvan@mik.pte.hu

Progress in Content

Transfer Functions

Transfer function (def.1) (network function) H(s)

- ❑ The ratio of the output response Y(s) to the input excitation X(s)
- ❑ Initial conditions are zero.

$$H(s) = \frac{Y(s)}{X(s)}$$

Four possible transfer functions

- ❑ Voltage gain → $H(s) = \frac{V_o(s)}{V_i(s)}$
- ❑ Current gain → $H(s) = \frac{I_o(s)}{I_i(s)}$
- ❑ Transfer impedance → $H(s) = \frac{V_o(s)}{I_i(s)}$
- ❑ Transfer admittance → $H(s) = \frac{I_o(s)}{V_i(s)}$

$$Y(s) = H(s)X(s)$$

$$x(t) = \delta(t) \rightarrow X(s) = 1 \rightarrow Y(s) = H(s) \rightarrow y(t) = h(t)$$

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

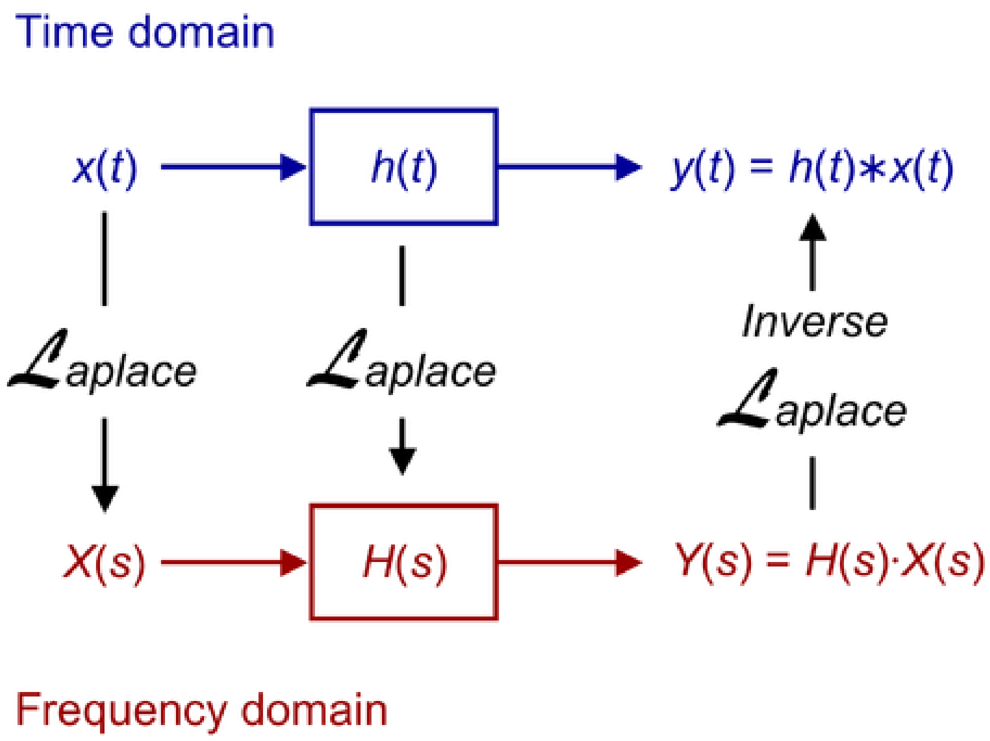
Transfer function (def.2) (network function) H(s)

- ❑ The Laplace transform of the $h(t)$ unit impulse response of a system *(súlyfüggvény)*

System response for <u>ANY</u> input excitation by…

- ❑ Using $H(s)$ in the s-domain
- ❑ Convolution integral in time domain *(⟶ next)*

Convolution Integral vs. Laplace Transform



Laplace transform use cases

- ❑ Solving integro-differential equations
- ❑ General LTI circuit analysis
- ❑ Obtaining transfer function
- ❑ Signal processing, control systems
- ❑ Network stability analysis
- ❑ Network synthesis

Progress in Content

The Convolution Integral

Convolution of 2 signals *(tekeredés)*

- Time-reversing value (one is shifted-product-integral)
- Useful tool for characterizing physical systems

$$y(t) \triangleq \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = x(t) * h(t)$$

- $x(t)$ excitation
- $y(t)$ response of the system
- $h(t)$ unit impulse response of the system *(súlyfüggv.)*
- λ dummy variable
- $*$ convolution (denote)

$$y(t) = \int_{0}^{\infty} x(\lambda)h(t-\lambda)d\lambda \ \ if\ x(t) = 0\ for\ t < 0$$

Properties

- Commutative $y(t) = x(t) * h(t) = h(t) * x(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

- Distributive $f(t) * [x(t) + y(t)] = f(t) * x(t) + f(t) * y(t)$
- Associative $f(t) * [x(t) * y(t)] = [f(t) * x(t)] * y(t)$
- $f(t) * \delta(t) = f(t)$
- $f(t) * \delta(t-t_0) = f(t-t_0)$
- $f(t) * \delta'(t) = f'(t)$
- $f(t) * u(t) = \int_{-\infty}^{\infty} f(\lambda)d\lambda$

Link bw. Laplace Transform and The Convolution Integral

$$f(t) = f_1(t) * f_2(t) = \int_0^{\infty} f_1(\lambda) f_2(t-\lambda)\,d\lambda \rightarrow F(s) = \mathcal{L}[f_1(t) * f_2(t)] = F_1(s) \cdot F_2(s)$$ (←w/o proving)

Convolution in time domain is equivalent to multiplication in s-domain

- ❑ Evaluating convolution integral in time domain … *rather difficult* → (inverse) Laplace transform

Example $x(t) = 4e^{-t}, h(t) = 5e^{-2t} \rightarrow x_1(t) * h(t) = ?$

$$h(t) * x(t) = \mathcal{L}^{-1}[H(s) \cdot X(s)] = \mathcal{L}^{-1}\left[\left(\frac{5}{s+2}\right)\left(\frac{4}{s+1}\right)\right] = \mathcal{L}^{-1}\left[\frac{20}{s+1} + \frac{-20}{s+2}\right] = 20(e^{-t} - e^{-2t}), t \geq 0$$

- ❑ Evaluating convolution integral from graphical point of view … *more convenient* (→ next)

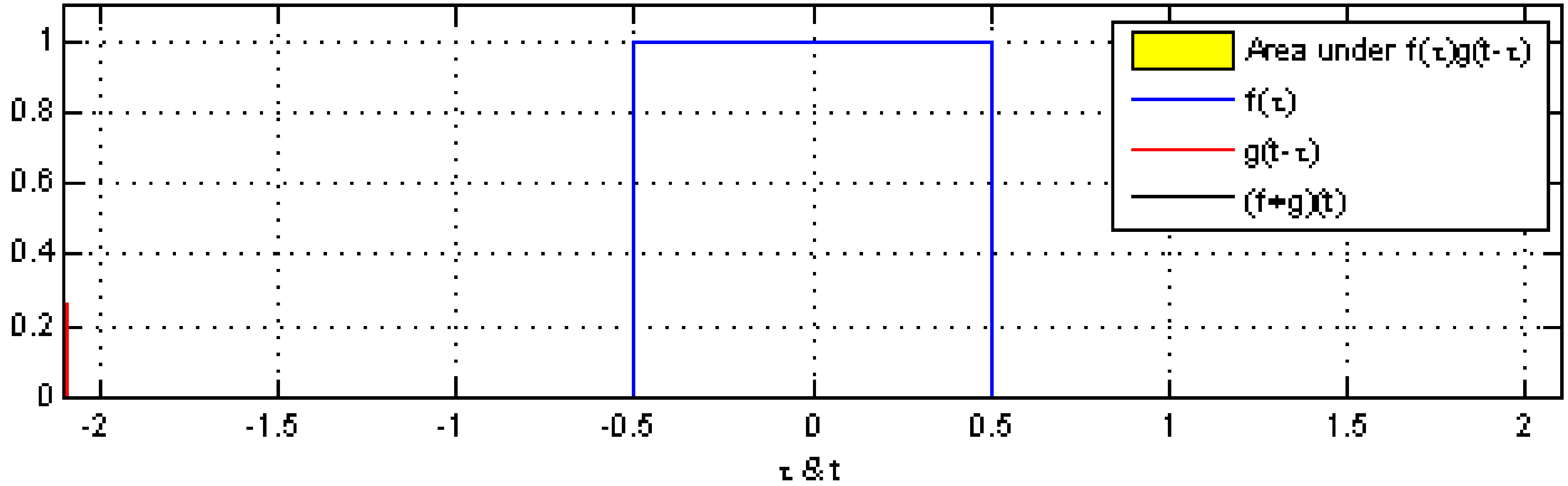
gyurcsek.istvan@mik.pte.hu

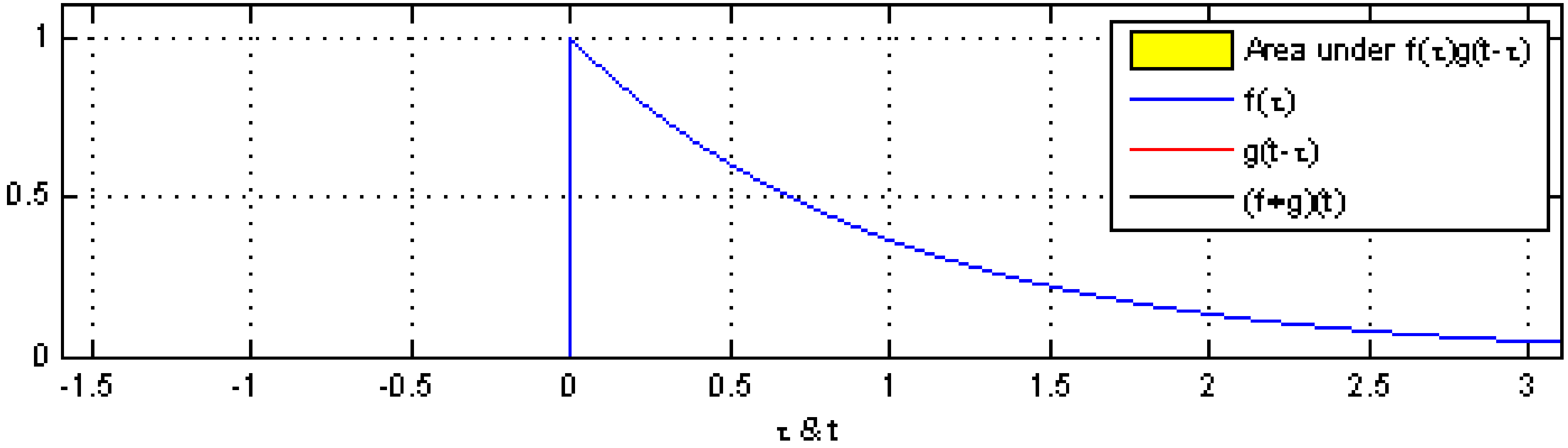
Graphical Illustration for Convolution Integral

$$(f * g)(t) =$$

$$f(t) * g(t) =$$

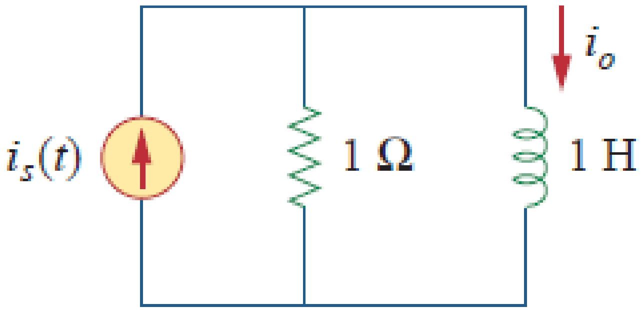
$$\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$





Convolution Integral for Circuit Analysis – Example

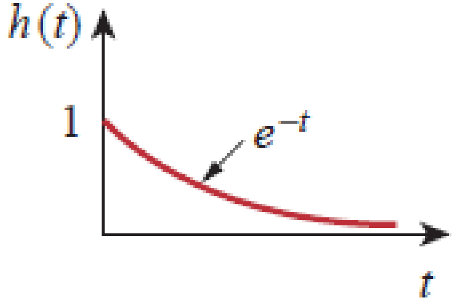
Use the convolution integral to find the $i_0(t)$ response due to the given excitation.

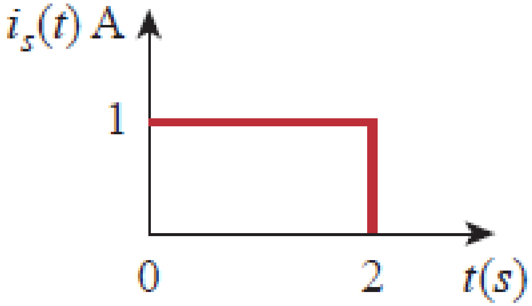


(1) Find $h(t)$ unit impulse response *(súlyfüggvény)*

$$current\ div. \rightarrow I_0 = \frac{1}{s+1} I_S \rightarrow H(s) = \frac{I_0}{I_S} = \frac{1}{s+1} \rightarrow h(t) = e^{-t}u(t)$$

(2) Find $i_0(t)$ using convolution integral.





$$I_0(s) = H(s)I_S(s) \rightarrow i_0(t) = h(t) * i_S(t)$$

$$i_S(t) = u(t) - u(t-2)$$

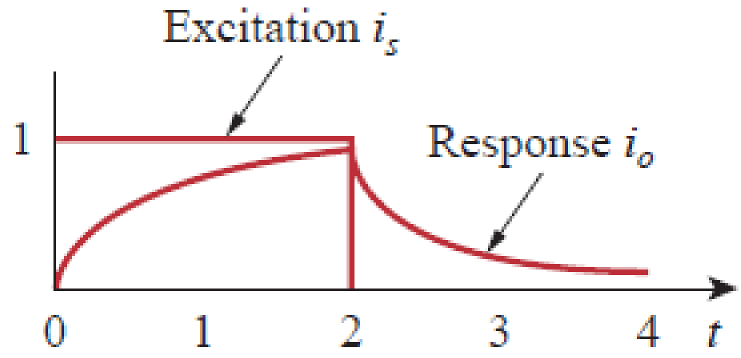
$$i_0(t) = \int_0^t i_S(\lambda)h(t-\lambda)d\lambda = \int_0^t [u(\lambda) - u(\lambda-2)\,]e^{-(t-\lambda)}d\lambda\ \ (\rightarrow next)$$

Convolution Integral Example

$$i_0(t) = \int_0^t [u(\lambda) - u(\lambda - 2)]e^{-(t-\lambda)}d\lambda$$

$$0 < t < 2 \rightarrow i_0'(t) = \int_0^t 1 \cdot e^{-(t-\lambda)}d\lambda = e^{-t}\int_0^t e^{\lambda}d\lambda = e^{-t}(e^t - 1) = 1 - e^{-t}$$

$$t > 2 \rightarrow i_0''(t) = \int_2^t 1 \cdot e^{-(t-\lambda)}d\lambda = e^{-t}\int_2^t e^{\lambda}d\lambda = e^{-t}(e^t - e^2) = 1 - e^2e^{-t}$$



$$i_0(t) = i_0'(t) - i_0''(t) = (1 - e^{-t})[u(t-2) - u(t)] - (1 - e^2e^{-t})u(t-2) = \begin{cases} 1 - e^{-t}\ A & 0 < t < 2 \\ (e^2 - 1)e^{-t}\ A & t > 2 \end{cases}$$

Progress in Content

Table of Laplace Transforms 1

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$

Table of Laplace Transforms 2

15.	$\sin(at+b)$	$\dfrac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\dfrac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$

Table of Laplace Transforms 3

25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Questions