DR. GYURCSEK ISTVÁN

Fourier Transform

Sources and additional materials (recommended)

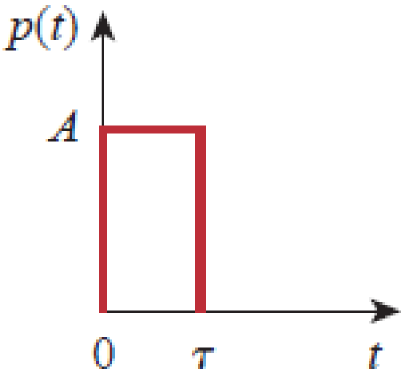
- *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- *Simonyi K.: Villamosságtan. AK Budapest 1983, ISBN:9630534134*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*

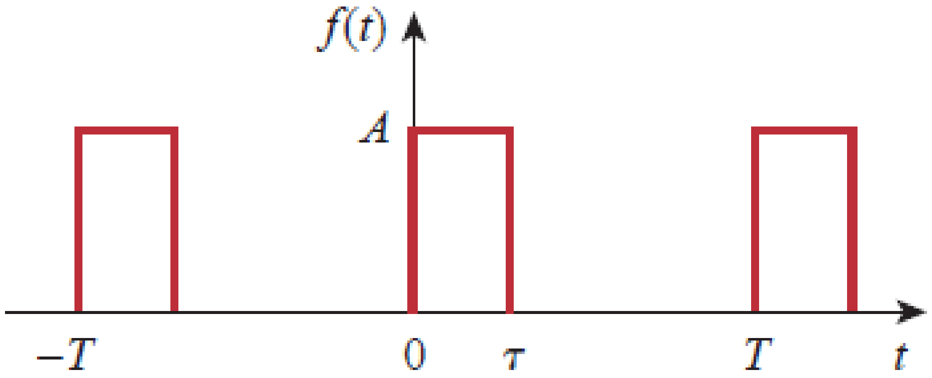
- **Definition of the Fourier Transform**
- Properties of the Fourier Transform
- Parseval’s Theorem for Nonperiodic Funcs
- Comparing Fourier and Laplace Transforms
- Applications of Fourier Transform

Introduction

- **Laplace transform** → useful analyzing tool for $t > 0$ with known initial conditions
- **Fourier transform** → useful analyzing tool for $-\infty < t < \infty$ with NO initial conditions

- Fourier series → Discrete Fourier analysis → frequency spectrum of periodic functions
- Fourier transform → Continuous Fourier analysis → frequency spectrum of nonperiodic functions
  (idea: nonperiodic function = periodic function w. infinite period)



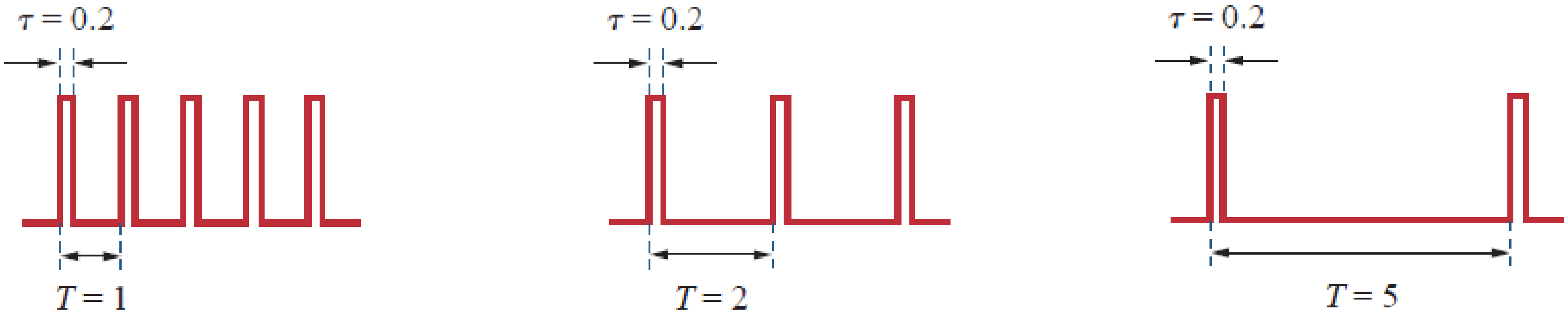


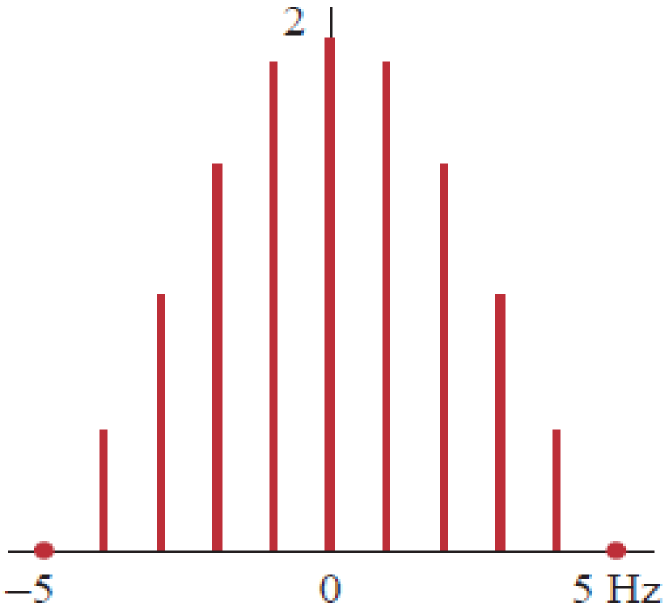
$p(t) = nonperiodic\ func$ $\qquad f(t) = f(t + nT) = periodic\ func$

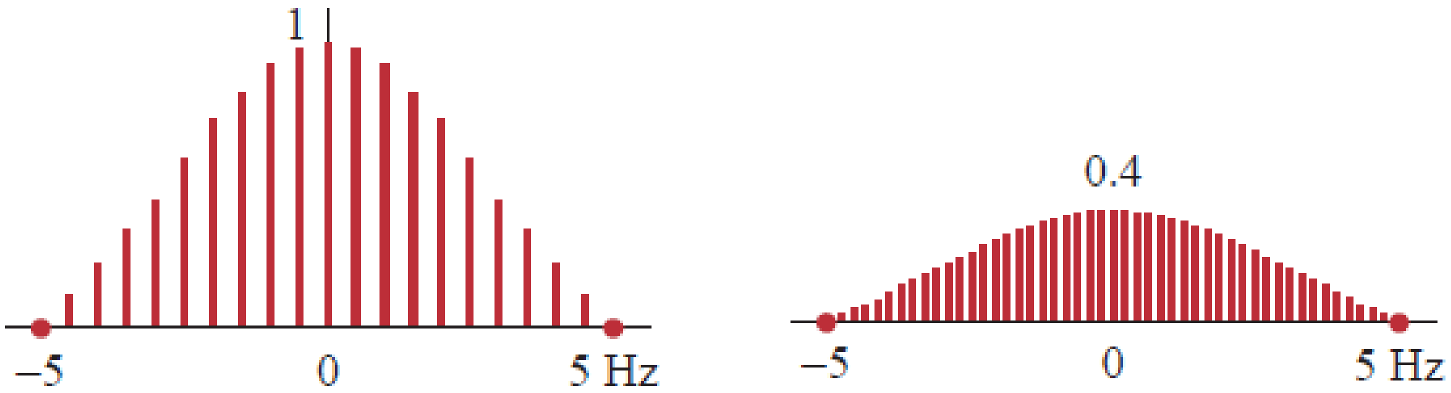
$T \to \infty\ \ldots\ f(t) \to p(t) = nonperiodic\ func$

$\Delta\omega \to 0 \ldots discrete\ spectrum \to continuous\ spectrum$

Effect of Increasing T on the Spectrum







Impulse trains with increasing *T* → effect on sectrum

Source: *L.Balmer, Signals and Systems, An Introduction. Prentice-Hall, London, 1991, p.229.*

Fourier Transform from Fourier Series

$$f(t)=\sum_{n=-\infty}^{\infty}\boldsymbol{D}_n e^{jn\omega_0 t} \qquad \boldsymbol{D}_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)\,e^{-jn\omega_0 t}dt$$

Space bw. harmonics → $\Delta\omega=(n+1)\omega_0-n\omega_0=\omega_0=\frac{2\pi}{T}$

$$f(t)=\sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f(t)\,e^{-jn\omega_0 t}dt\right]e^{jn\omega_0 t}=\sum_{n=-\infty}^{\infty}\left[\frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)\,e^{-jn\omega_0 t}dt\right]e^{jn\omega_0 t}$$

$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f(t)\,e^{-jn\omega_0 t}dt\right]\Delta\omega e^{jn\omega_0 t}$$

If $T\to\infty$ *then* $\sum_{n=-\infty}^{\infty}\to\int_{-\infty}^{\infty}$, $\quad\Delta\omega\to d\omega,\quad n\omega_0\to\omega$

$$\to f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right]e^{j\omega t}d\omega\to F(\omega)=\mathcal{F}[f(t)]=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$f(t)=\mathcal{F}^{-1}[F(\omega t]=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)\,e^{j\omega t}d\omega$$

Convergence condition (sufficient)

$$\int_{-\infty}^{\infty}|f(t)|\,dt<\infty$$

gyurcsek.istvan@mik.pte.hu

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Properties

(1) Linearity $\mathcal{F}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(j\omega) + a_2 F_2(j\omega)$

(2) Scaling $\mathcal{F}[f(at)] = \frac{1}{|a|} F(\frac{j\omega}{a})$

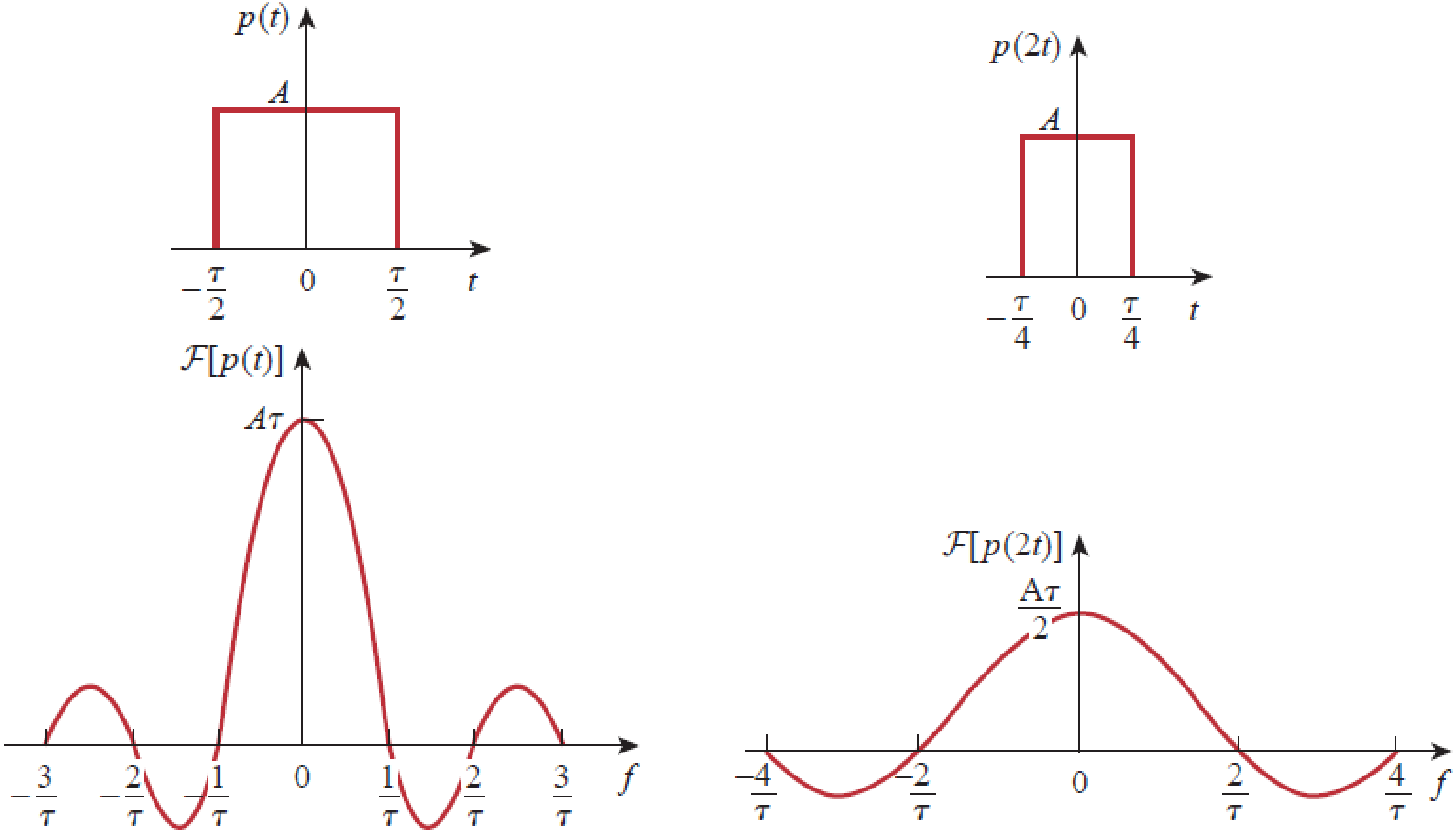
$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} dt$$

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-x\left(\frac{j\omega}{a}\right)} \frac{dx}{a} \leftarrow x = a \cdot t \qquad \dots = \frac{1}{a} \int_{0}^{\infty} f(x) e^{-x(\frac{s}{a})} dx = \frac{1}{a} F(\frac{j\omega}{a})$$

Scaling…

- ❑ Time expansion → frequency compression
- ❑ Time compression → frequency expansion
- ❑ *(see example next …)*

Time Scaling Example with ‘sinc’



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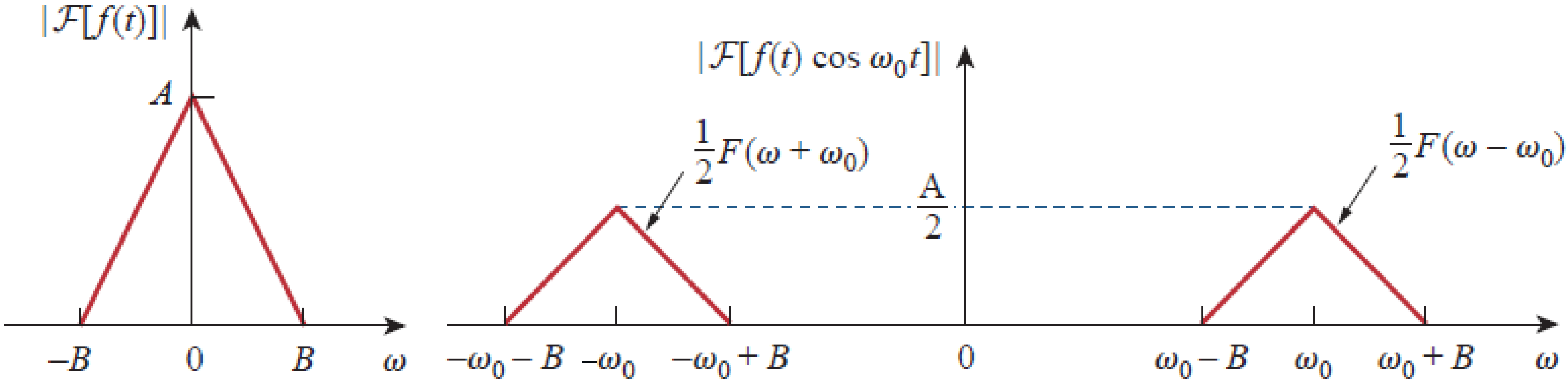
Properties (cont)

(3) Time shift $\mathcal{F}[f(t-t_0)] = e^{-j\omega t_0}F(\omega)$

(4) Frequency shift (amplitude modulation) $\mathcal{F}\left[f(t)e^{j\omega_0 t}\right] = F(\omega-\omega_0)$

Example



Properties (cont)

(5) Time differentiation $\mathcal{F}[f'(t)] = j\omega F(\omega)$

(6) Time integration $\mathcal{F}\left[\int_{-\infty}^{t} f(t)dt\right] = \dfrac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$

(7) Reversal $F(\omega) = \mathcal{F}[f(t)\,] \rightarrow \mathcal{F}[f(-t)\,] = F(-\omega) = F^{*}(\omega)$

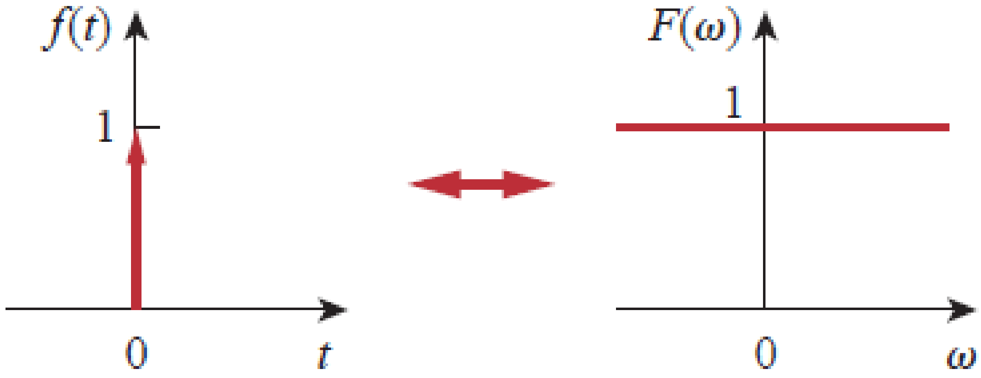
(8) Duality $F(\omega) = \mathcal{F}[f(t)\,] \rightarrow \mathcal{F}[F(t)\,] = 2\pi f(-\omega)$

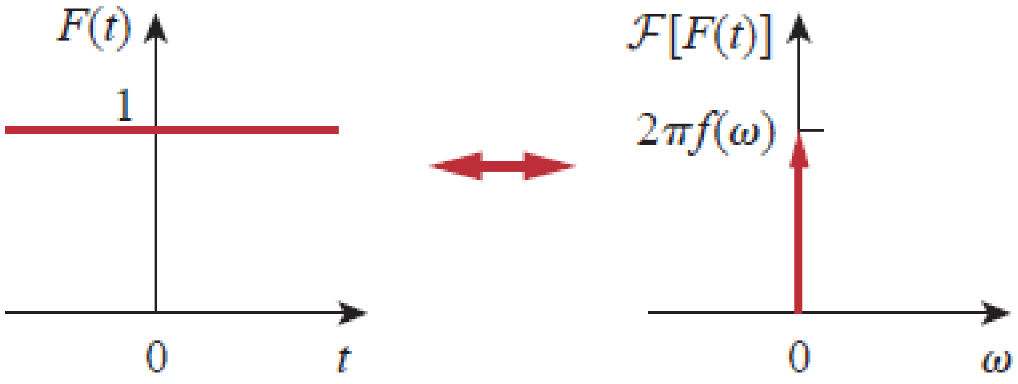
(9) Convolution

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

$$Y(\omega) = \mathcal{F}[h(t) * x(t)] = H(\omega) \cdot X(\omega)$$

Duality examples – transform of pulse and dc level





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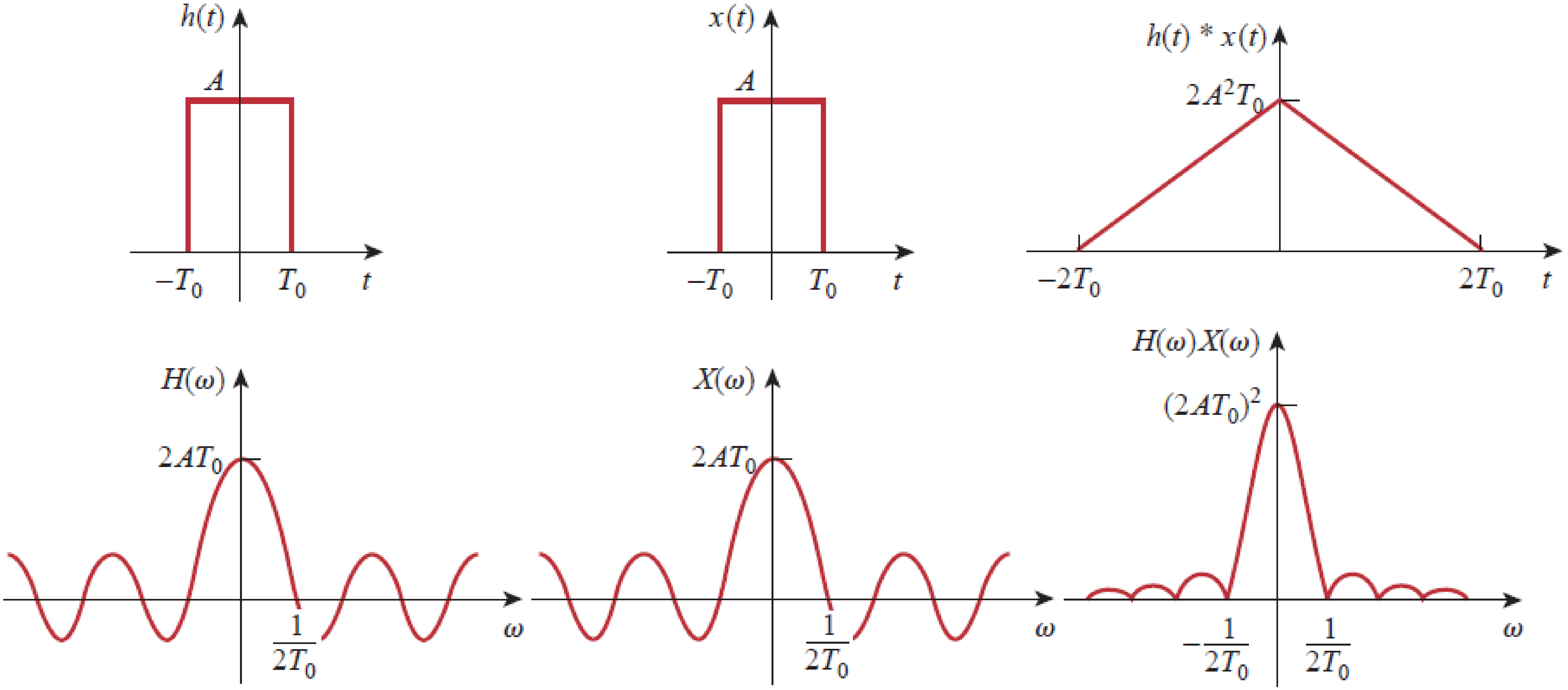
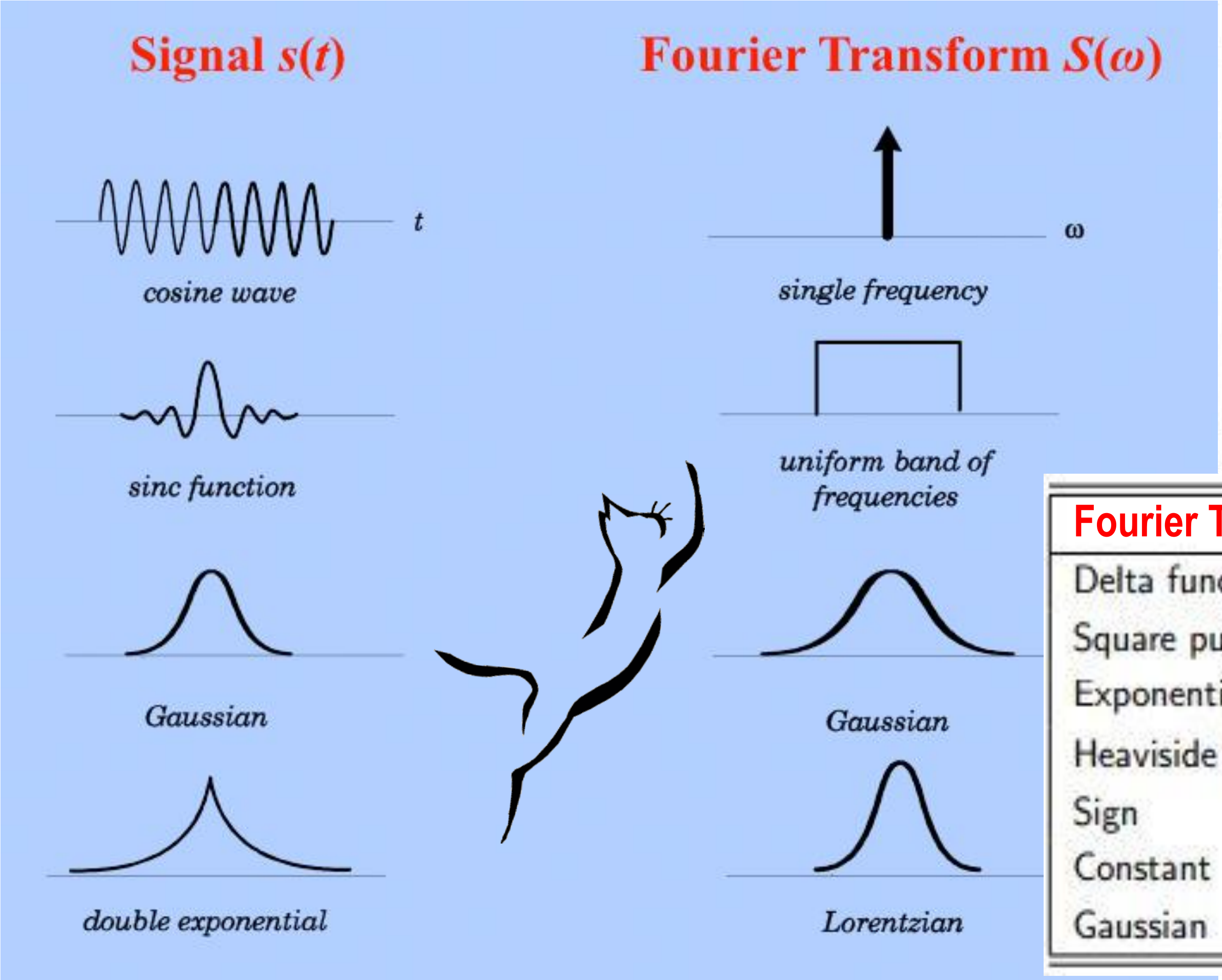
Convolution Illustration Example

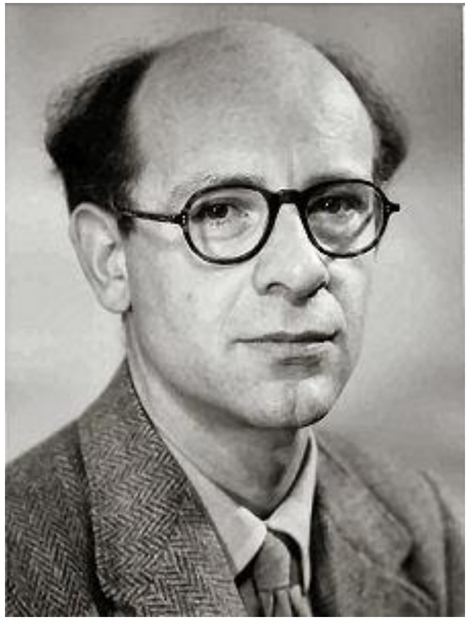


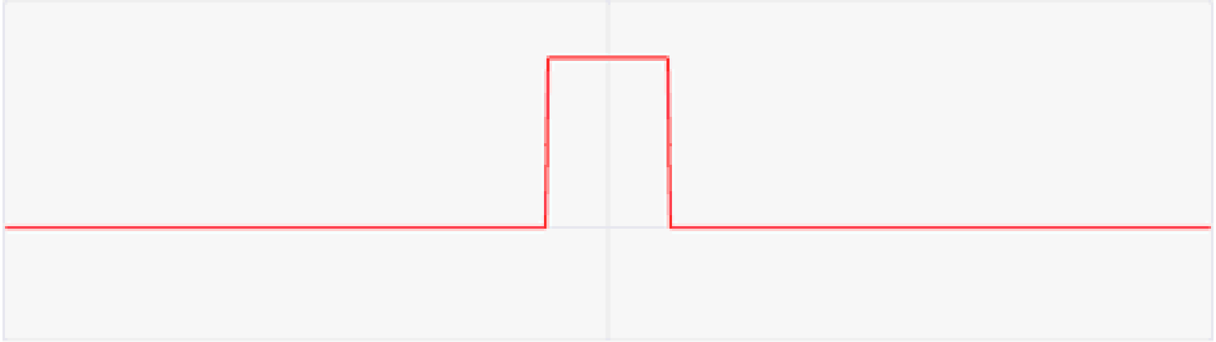
Fourier Transform Examples



Fourier Table...	$f(x)$	$\hat{f}(k)$
Delta function	$\delta(x)$	1
Square pulse	$H(a-\lvert x\rvert)$	$\frac{1}{\pi k}\sin 2\pi ka$
Exponential	$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+(2\pi k)^2}$
Heaviside function	$H(x)$	$\frac{1}{2}\delta(k)+\frac{1}{2\pi ik}$
Sign	$H(x)-H(-x)$	$\frac{1}{\pi ik}$
Constant	1	$\delta(k)$
Gaussian	$e^{-\pi x^2}$	$e^{-\pi k^2}$

Comment for Sinus Cardinal





- Sinc = sinus cardinalis (Lat), cardinal sine (Eng)
- P. M. Woodward (1952): Information theory and inverse probability in telecommunication →
  „ ... occurs so often in Fourier analysis ... merit some notation of its own"

$$f(x)$$

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Parseval's Theorem for Non-Periodic Functions

$$W = \int_{-\infty}^{\infty} p(t)\,dt \qquad R = 1\,\Omega \rightarrow p(t) = v^2(t) = i^2(t) = f^2(t) \qquad W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

Parseval's theorem → total energy delivered to a 1 Ω resistor equals

- ❑ Total area under $f^2(t)$ or
- ❑ $1/2\pi$ times the total area under $(\mathcal{F}[f(t)])^2$

Comment: *$|F(\omega)|^2$ = energy spectral density of $f(t)$*

Marc-Antoine Parseval
French mathematician
(1755 - 1836)

Parseval's theorem (in general)

- ❑ Energy of nonperiodic signals → spread over the entire spectrum
- ❑ Energy of periodic signals → concentrated at the harmonic components

Energy spectral density → even function $\qquad W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{\pi}\int_{0}^{\infty} |F(\omega)|^2\,d\omega$

Energy delivered to a 1 Ω resistor in a freq. band $\qquad W_{1\Omega} = \frac{1}{\pi}\int_{\omega 1}^{\omega 2} |F(\omega)|^2\,d\omega$

- ❑ Definition of the Fourier Transform
- ❑ Properties of the Fourier Transform
- ❑ Parseval’s Theorem for Nonperiodic Funcs
- ❑ **Comparing Fourier and Laplace Transforms**
- ❑ Applications of Fourier Transform

Comparing the Fourier and Laplace Transforms

LAPLACE TRANSFORM	FOURIER TRANSFORM
One-sided integral → useful for positive-time functions	Double-sided integral → applicable for all time functions
For positive-time functions if $\int_0^\infty \lvert f(t) \rvert dt < \infty \rightarrow F(\omega) = F(s)\rvert_{s=j\omega}$	
Related to entire s plain	Restricted to jω axis
Wider range of functions (i.e. u(t) has Laplace transform)	Lower range of functions (i.e. u(t) has no Fourier transform)
Initial conditions → better for transient analysis	No initial conditions → useful for steady state analysis
Theoretical s plain	Better view into frequency characteristics

- ❑ Definition of the Fourier Transform
- ❑ Properties of the Fourier Transform
- ❑ Parseval’s Theorem for Nonperiodic Funcs
- ❑ Comparing Fourier and Laplace Transforms
- ❑ **Applications of Fourier Transform**

Applications

Jean-Baptiste Joseph Fourier

$$x(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega \cdot t} dt$$

TAILOR

Greenhouse effect
Fourier transform
Fourier series

BARON

COMPUTER SCIENCE

ACOUSTICS

SPECTROSCOPY

OPTICS (1) FFT

(2) CIRCUIT ANALYSIS

ELECTRICAL ENGINEERING

SIGNAL PROCESSING → (3) SAMPLING

COMMUNICATIONS SYSTEMS → (4) AMPLITUDE MODULATION (AM)

Circuit Applications

Fourier transform

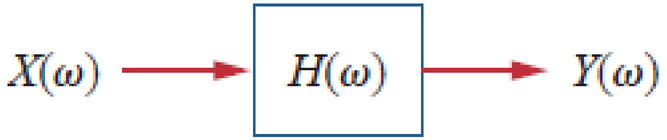
- ❑ Generalized phasor technique to nonperiodic functions
- ❑ *'Ohm's law'* is still valid $V(\omega) = Z(\omega) \cdot I(\omega)$

Element	Time Domain	Frequency Domain
R	$v = R \cdot i$	$V(\omega) = R \cdot I(\omega)$
L	$v = L \cdot \frac{di}{dt}$	$V(\omega) = j \cdot \omega \cdot L \cdot I(\omega)$
C	$i = C \cdot \frac{dv}{dt}$	$V(\omega) = \frac{1}{j \cdot \omega \cdot C} \cdot I(\omega)$

KCL + KVL → same circuit techniques

- ❑ Voltage division, current division
- ❑ Nodal analysis, mesh analysis
- ❑ Superposition theorem, source transform
- ❑ Thevenin's theorem, Norton's theorem

General response w. transfer function



$$Y(\omega) = H(\omega) \cdot X(\omega)$$

$$x(t) = \delta(t) \rightarrow X(\omega) = 1 \rightarrow Y(\omega) = H(\omega) = \mathcal{F}[h(t)]$$

Transfer function = Fourier transform of impulse response !

gyurcsek.istvan@mik.pte.hu

App. – AM 1

Amplitude modulation (AM)
→ carrier amplitude is controlled by the modulating signal.

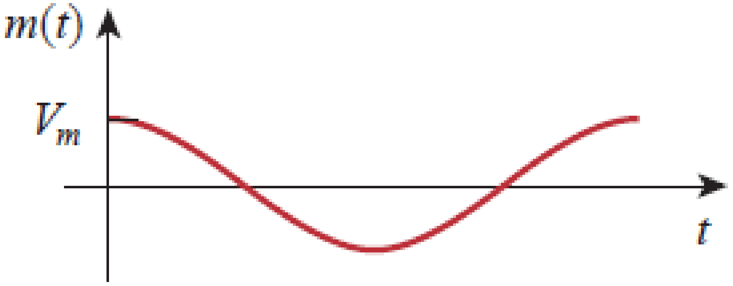
$$m(t) = V_m \cos\omega_m t \ , \qquad c(t) = V_c \cos\omega_c t \ , \qquad \omega_c \gg \omega_c$$

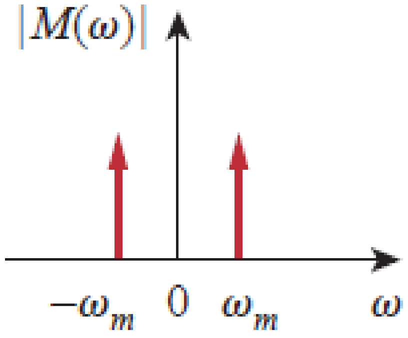
$$f(t) = V_c[1 + m(t)]\cos\omega_c t$$

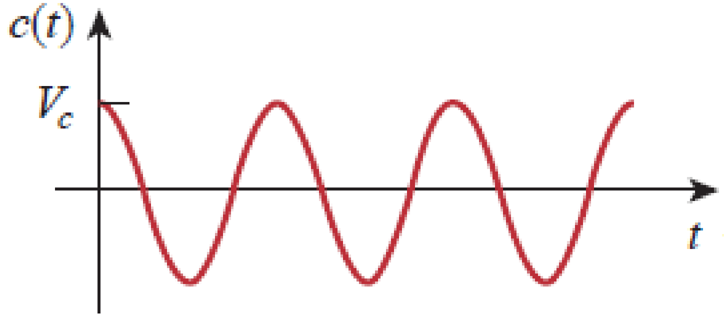
$$F(\omega) = \mathcal{F}[V_c \cos\omega_c t] + \mathcal{F}[V_c m(t)\cos\omega_c t]$$

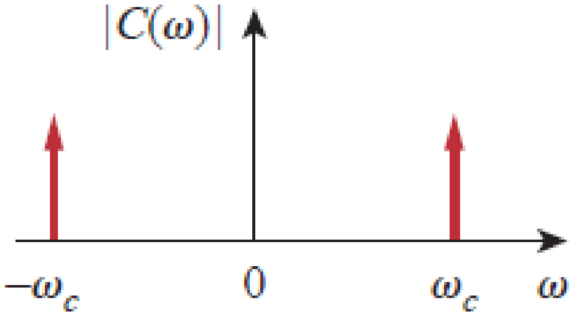
$$= V_c\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$$

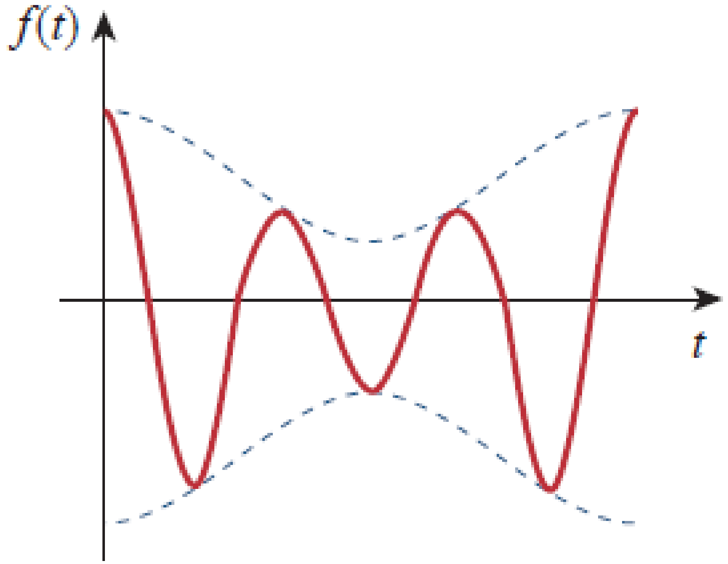
$$+ \frac{V_c}{2}[M(\omega - \omega_c) + M(\omega + \omega_c)]$$

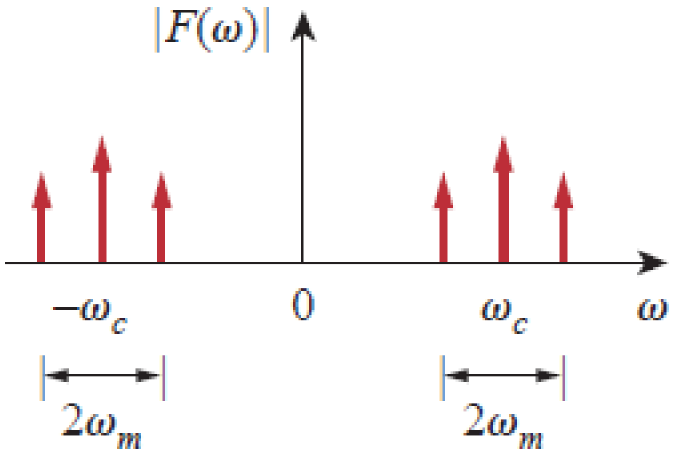












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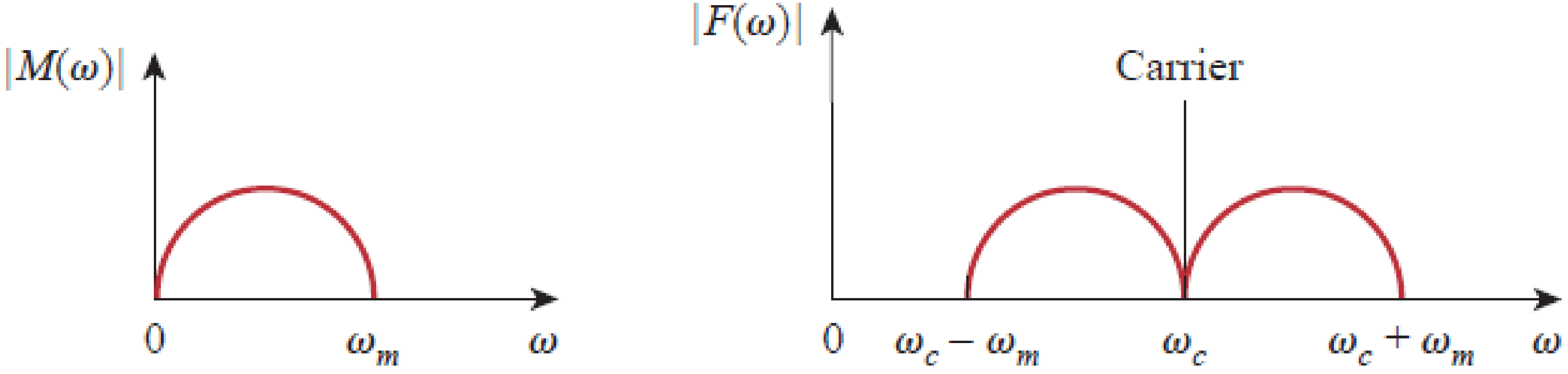
App. – AM 2

TX → $m(t)$ → band-limited non-sinusoidal signal



RX → info recovering from the modulated carrier *(demodulation)*

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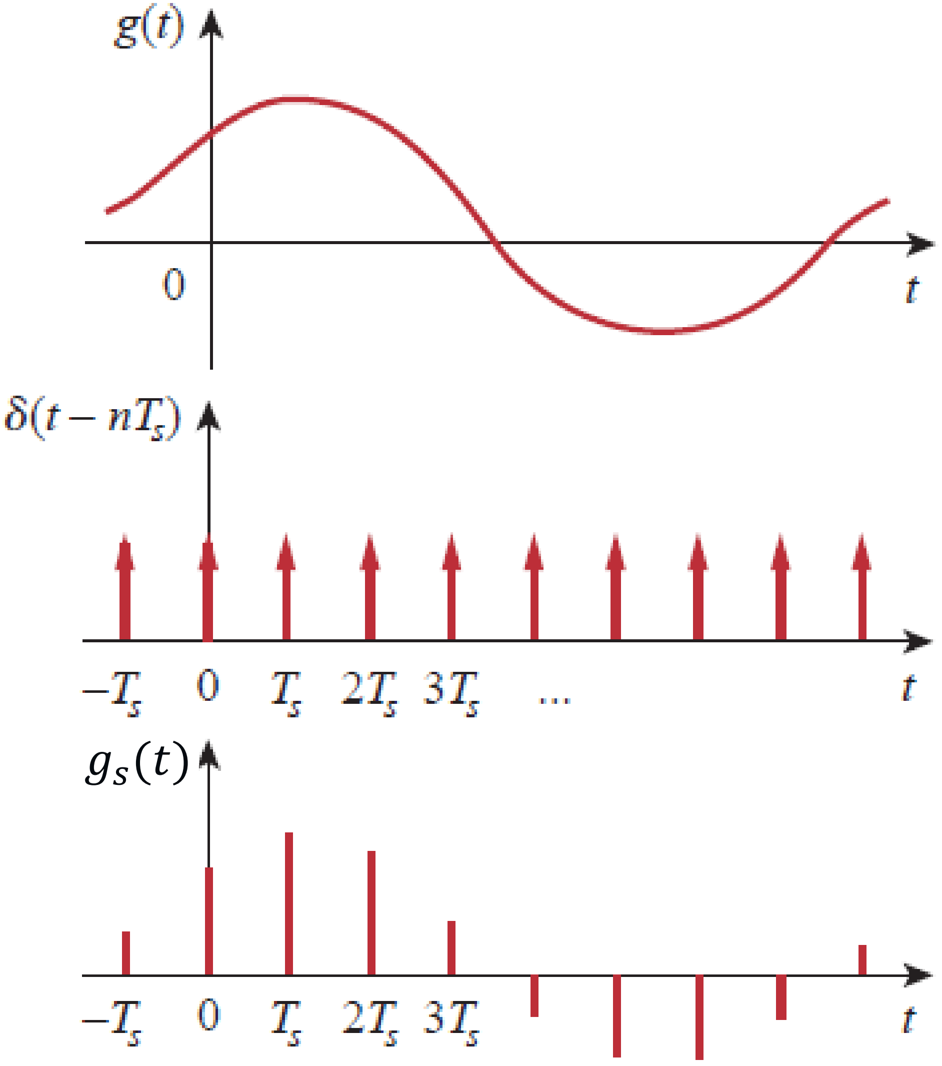
App. – Sampling



$g(t)$ → analog singal; $\delta(t-nT_S)$ → train of impulses

T_S → sampling interval; $f_s = {}^{1}/_{T_S}$ → sampling rate;

$g_s(t)$ → sampled signal *(for digital processing)*

$$g_s(t) = g(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_S) = \sum_{n=-\infty}^{\infty} g(nT_S)\delta(t-nT_S)$$

$$G_s(\omega) = \sum_{n=-\infty}^{\infty} g(nT_S)\mathcal{F}[\delta(t-nT_S)] = \sum_{n=-\infty}^{\infty} g(nT_S)e^{-jn\omega T_S}$$

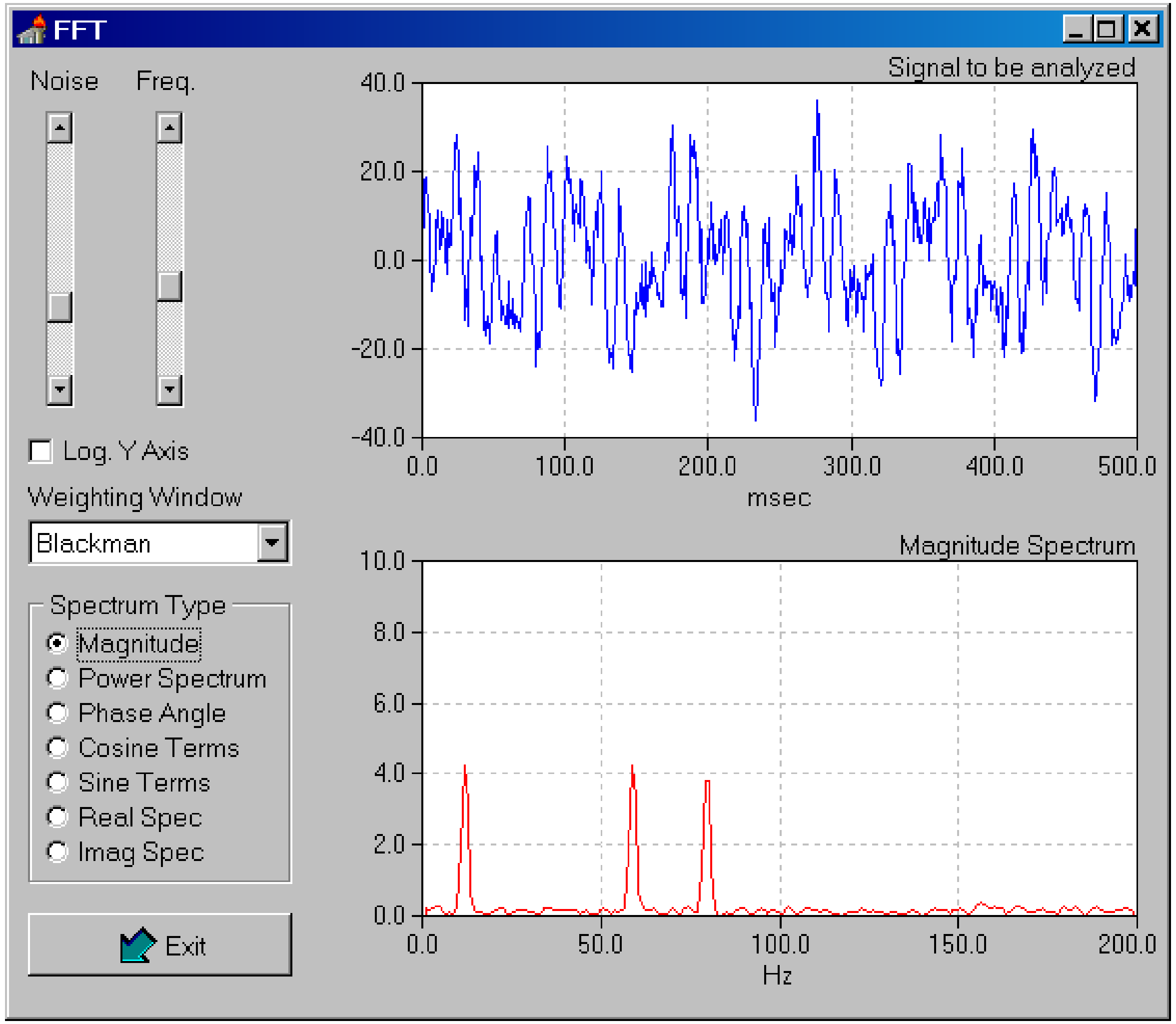
$$G_s(\omega) = \frac{1}{T_S}\sum_{n=-\infty}^{\infty} G(\omega+n\omega_S) \leftarrow \omega_S = \frac{2\pi}{T_S}$$

Nyquist-Shannon sampling theorem *(equivalent part of sampling theorem)*
f_H band limited signal may be completely recovered from samples if $f_S \geq 2f_H$
(→ *no overlapping*!)

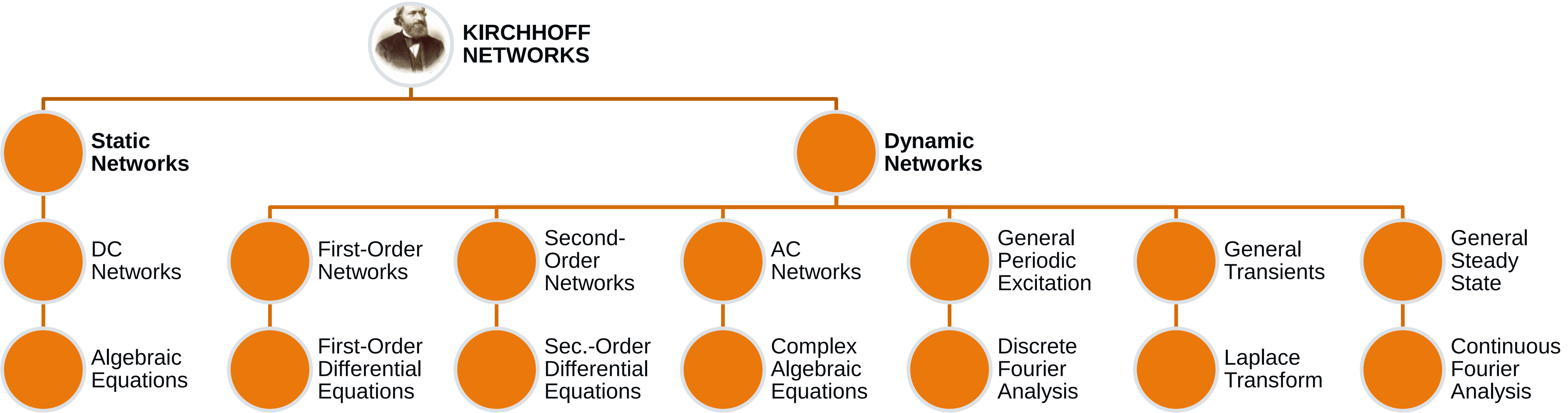
gyurcsek.istvan@mik.pte.hu

App. – FFT

- **DFT Algorithm** *(DFT = Discrete Fourier Transform)*
  - Rapid!
  - $f(t) \rightarrow DFT\ ,\ DFT \rightarrow f(t)$
- **Method**
  - Factorizing the DFT matrix into a product of sparse (mostly zero) factors.
- **Basic ideas**
  - 1965, (some algorithms; 1805, Gauss)
- **Importance**
  - In Top-10 Algorithms of 20th Century *(IEEE: Institute of Electrical and Electronics Engineers)*



Kirchhoff Network Analysis



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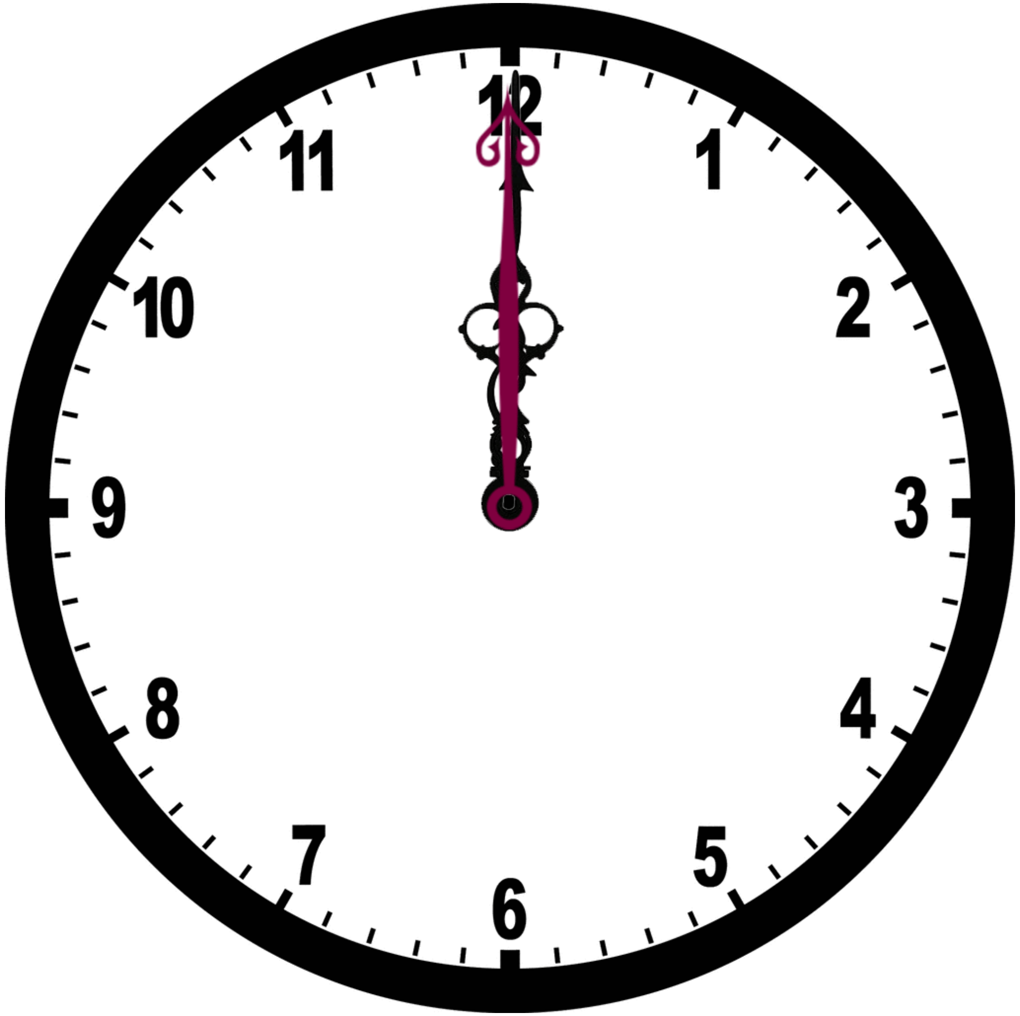
Questions



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