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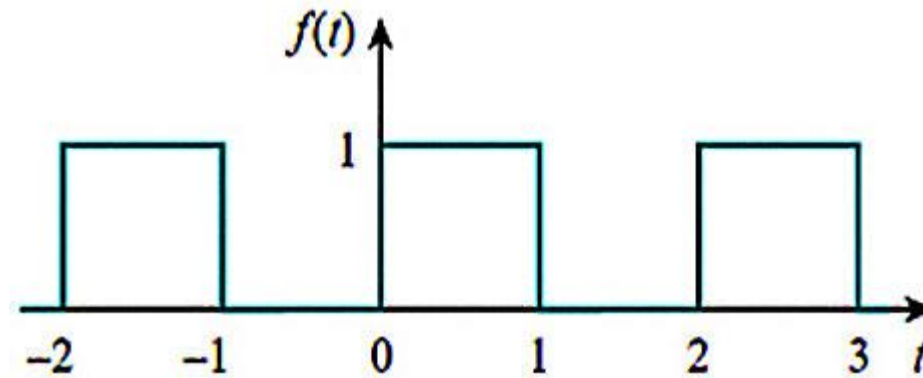
Fourier Series - Examples

Sources and additional materials (recommended)

- ❑ *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- ❑ *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- ❑ *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- ❑ *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*

FRS.01

Determine the Fourier series of the waveform.
Obtain the amplitude and phase spectra.



Solution:

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \quad \omega_0 = \frac{2\pi}{T} = \pi$$

$$A_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2} \left(\int_0^1 1 dt + \int_1^2 0 dt \right) = \frac{1}{2}$$



$$A_k = \frac{2}{T} \int_0^T f(t) \cos k\omega_0 t dt = \frac{2}{2} \left(\int_0^1 1 \cos k\pi t dt + \int_1^2 0 \cos k\pi t dt \right) = \frac{1}{k\pi} \sin k\pi = 0$$

$$B_k = \frac{2}{T} \int_0^T f(t) \sin k\omega_0 t dt = \frac{2}{2} \left(\int_0^1 1 \sin k\pi t dt + \int_1^2 0 \sin k\pi t dt \right) = \frac{-\cos k\pi t}{k\pi} \Big|_0^1 = \frac{1}{k\pi} (-\cos k\pi + 1)$$

$$B_k = \begin{cases} 2/k\pi & k = \text{odd (PN)} \\ 0 & k = \text{even (PS)} \end{cases} \quad f(t) = \frac{1}{2} + \frac{2}{\pi} \sin \pi t + \frac{2}{3\pi} \sin 3\pi t + \frac{2}{5\pi} \sin 5\pi t + \frac{2}{7\pi} \sin 7\pi t + \dots$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{i=1}^{\infty} \frac{1}{2i-1} \sin n\pi t, \quad n = 2i - 1$$

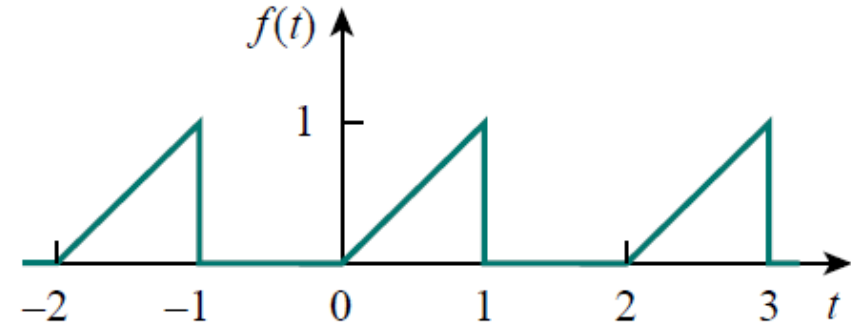
To obtain spectra: No $A_k \rightarrow C_k = B_k$ and $\varphi_k = 0$.

Fourier Analysis

FRS.02 – Obtain the Fourier series for the periodic function in Figure and **plot the amplitude and phase spectra**.

Solution: $f(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & 1 \leq t < 2 \end{cases}$ $\omega_0 = \frac{2\pi}{T} = \pi$

$$A_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2} \left(\int_0^1 t dt + \int_1^2 0 dt \right) = \frac{1}{2} \left. \frac{t^2}{2} \right|_0^1 = \frac{1}{4}$$



$$A_k = \frac{2}{T} \int_0^T f(t) \cos k\omega_0 t dt = \frac{2}{2} \left(\int_0^1 t \cos k\pi t dt + 0 \right) = \left[\frac{1}{k^2\pi^2} \cos k\pi t + \frac{1}{k\pi} \sin k\pi t \right] \Big|_0^1 = \frac{1}{k^2\pi^2} (\cos k\pi - 1) + 0 = \frac{(-1)^k - 1}{k^2\pi^2}$$

$$B_k = \frac{2}{T} \int_0^T f(t) \sin k\omega_0 t dt = \frac{2}{2} \left(\int_0^1 t \sin k\pi t dt + \int_1^2 0 \sin k\pi t dt \right) = \left[\frac{1}{k^2\pi^2} \sin k\pi t - \frac{t}{k\pi} \cos k\pi t \right] \Big|_0^1 = 0 - \frac{\cos k\pi}{k\pi} = \frac{(-1)^{k+1}}{k\pi}$$

$$f(t) = \frac{1}{4} + \sum_{i=1}^{\infty} \left[\frac{(-1)^k - 1}{(k\pi)^2} \cos k\pi t + \frac{(-1)^{k+1}}{k\pi} \sin k\pi t \right]$$

Fourier Analysis

$$f(t) = \frac{1}{4} + \sum_{i=1}^{\infty} \left[\frac{(-1)^k - 1}{(k\pi)^2} \cos k\pi t + \frac{(-1)^{k+1}}{k\pi} \sin k\pi t \right]$$

$$A_k = 0, B_k = -\frac{1}{k\pi} \leftarrow k = 2, 4, \dots$$

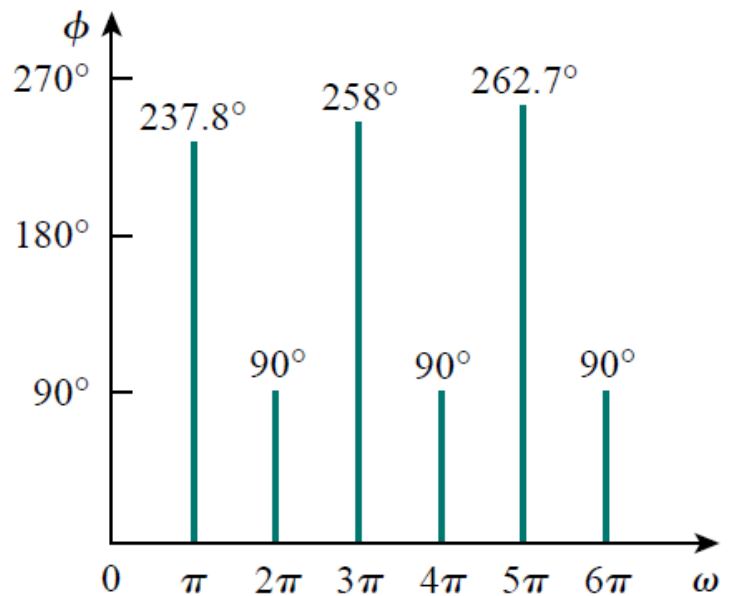
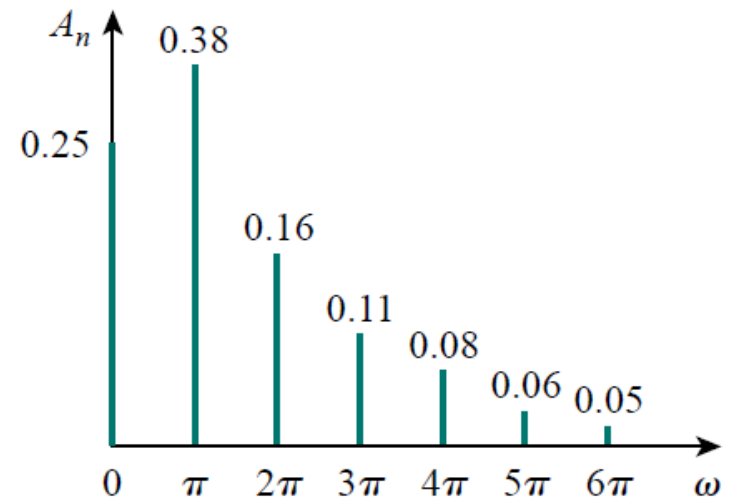
$$C_k = A_k - jB_k = 0 + j\frac{1}{k\pi} = \frac{1}{k\pi} e^{j90^\circ} \leftarrow k = 2, 4, \dots$$

$$A_k = -\frac{2}{(k\pi)^2}, B_k = \frac{1}{k\pi} \leftarrow k = 1, 3, \dots$$

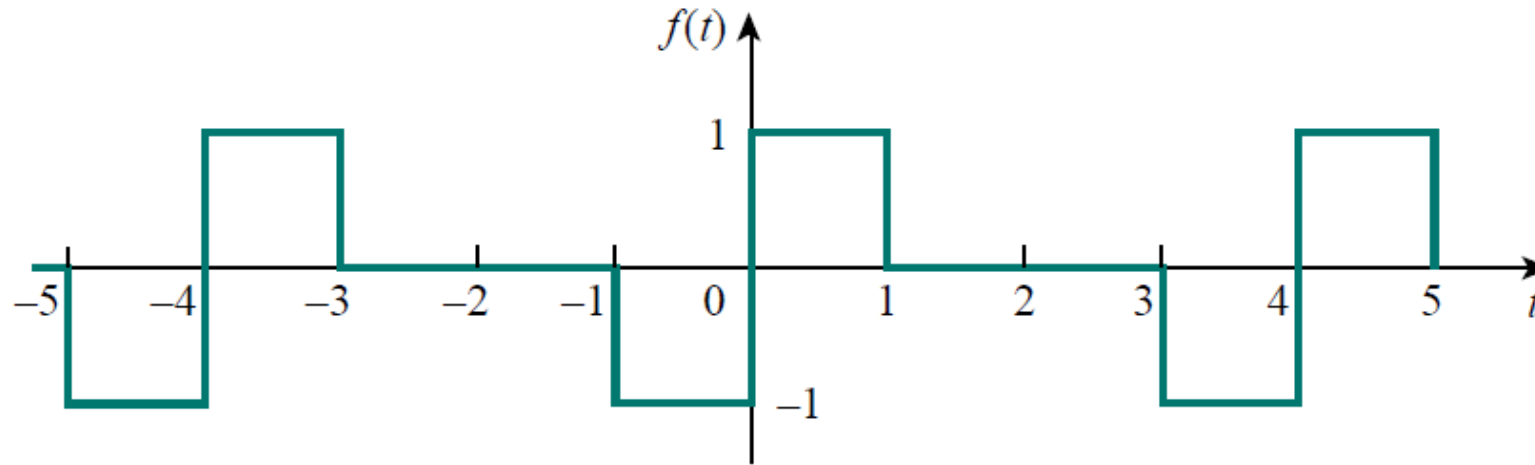
$$C_k = A_k - jB_k = -\frac{2}{(k\pi)^2} - j\frac{1}{k\pi} = C_k e^{j\varphi_k} \leftarrow k = 1, 3, \dots$$

$$C_k = \sqrt{A_k^2 + B_k^2} = \sqrt{\frac{4}{(k\pi)^4} + \frac{1}{(k\pi)^2}} = \frac{\sqrt{4 + (k\pi)^2}}{(k\pi)^2} \leftarrow k = 1, 3, \dots$$

$$\varphi_k = 180^\circ + \tan^{-1} \frac{k\pi}{2}$$



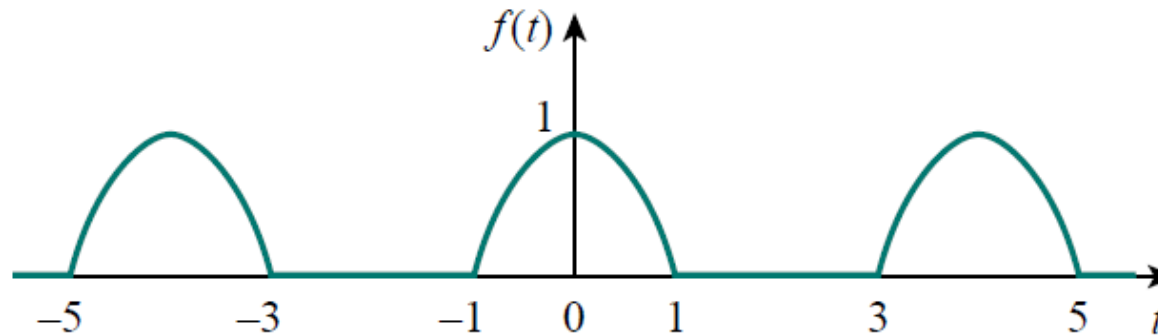
FRS.03 – Find the Fourier series expansion of $f(t)$ given in Figure.



Result:
$$f(t) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \left(1 - \cos \frac{k\pi}{2} \right) \sin \frac{k\pi}{2} t$$

FRS.04

Determine the Fourier series for the half-wave rectified cosine function shown in Figure..



Result:
$$f(t) = \frac{1}{\pi} + \frac{1}{2} \cos \frac{\pi}{2} t - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2 - 1} \cos k\pi t$$

FRS.05

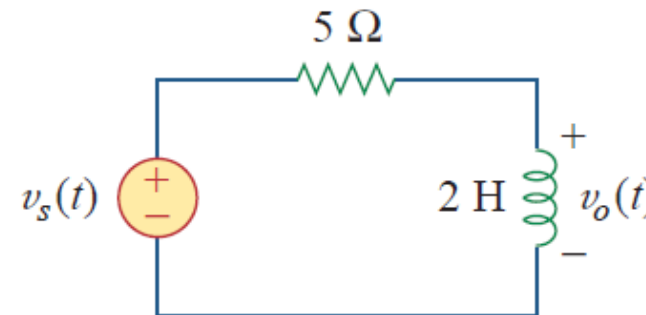
Find the $v_o(t)$ response voltage across the inductor in the circuit if the source voltage is

$$v_s(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{n} \sin n\pi t, \quad n = 2k - 1$$

Solution:

$$V_{0n} = V_{Sn} \frac{j\omega_n L}{R + j\omega_n L} = V_{Sn} \frac{j2n\pi}{5 + j2n\pi}$$

$$\omega_n = n \cdot \omega_1 = n \cdot \pi$$



$n = 0$

$$V_{S0} = \frac{1}{2} \rightarrow V_{00} = 0$$

n-th harmonic

$$v_S(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{n} \sin n\pi t, \quad n = 2k - 1$$

$$v_S(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{n} \cos(n\pi t - 90^\circ), \quad n = 2k - 1$$

$$V_{Sn} = \frac{2}{n\pi} e^{-j90^\circ} = \frac{-j2}{n\pi} \rightarrow V_{0n} = \frac{-j2}{n\pi} \frac{j2n\pi}{5 + j2n\pi} = \frac{4e^{-j \tan^{-1} \frac{2n\pi}{5}}}{\sqrt{25 + 4n^2\pi^2}}$$

$$v_0(t) = 0 + \sum_{k=1}^{\infty} \frac{4}{\sqrt{25 + 4n^2\pi^2}} \cos\left(n\pi t - \tan^{-1} \frac{2n\pi}{5}\right), \quad n = 2k - 1$$

$$v_0(t) = 0.50 \cos(\pi t - 51.5^\circ) + 0.2 \cos(3\pi t - 75^\circ) + 0.13 \cos(5\pi t - 81^\circ) + \dots$$

Fourier Analysis Applications

FRS.06

Determine the average power supplied to the circuit if $i(t) = 2 + 10 \cos(t + 10^\circ) + 6 \cos(3t + 45^\circ) \text{ A}$

Solution:
$$\mathbf{Z} = 10 \times \frac{1}{j2\omega} = \frac{10}{1 + j20\omega}$$

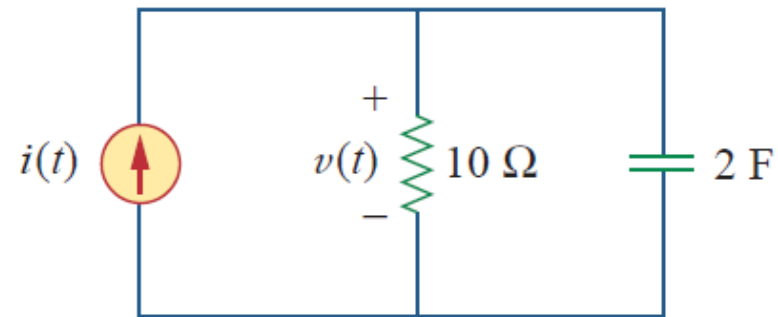
$$\mathbf{V} = \mathbf{I} \mathbf{Z} = \frac{10 \mathbf{I}}{1 + j20\omega} = \frac{10 \mathbf{I}}{\sqrt{1 + 400\omega^2} e^{j \tan^{-1} 20\omega}}$$

$$\omega = 0 \rightarrow I_0 = 2 \text{ A} \rightarrow V = 2 \cdot 10 = 20 \text{ V}$$

$$\omega = 1 \rightarrow I_1 = 10 e^{j10^\circ} \rightarrow V = \frac{10 \cdot 10 e^{j10^\circ}}{\sqrt{1 + 400} e^{j \tan^{-1} 20}} = 5 e^{-j77.14^\circ} \text{ V}$$

$$\omega = 3 \rightarrow I_3 = 6 e^{j45^\circ} \rightarrow V = \frac{10 \cdot 6 e^{j45^\circ}}{\sqrt{1 + 3600} e^{j \tan^{-1} 60}} = 1 e^{-j44.05^\circ} \text{ V}$$

$$v(t) = 20 + 5 \cos(t - 77.14^\circ) + 1 \cos(3t - 44.05^\circ) \text{ V}$$



Fourier Analysis Applications

$$i(t) = 2 + 10 \cos(t + 10^\circ) + 6 \cos(3t + 45^\circ) \text{ A}$$

$$v(t) = 20 + 5 \cos(t - 77.14^\circ) + 1 \cos(3t - 44.05^\circ) \text{ V}$$

Average power

$$\begin{aligned} P &= V_0 I_0 + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\varphi_{Un} - \varphi_{In}) = 20 \cdot 2 + \frac{5 \cdot 10}{2} \cos(-77.14^\circ - 10^\circ) + \frac{1 \cdot 6}{2} \cos(-44.05^\circ - 45^\circ) \\ &= 40 + 1.247 + 0.05 = 41.3 \text{ W} \end{aligned}$$

Alternatively

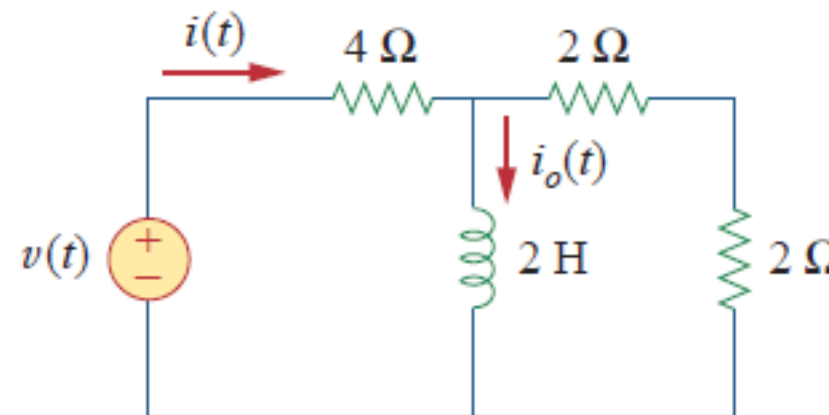
$$P = \frac{V_0^2}{R} + \frac{1}{2} \sum_i \frac{V_i^2}{R} = \frac{400}{10} + \frac{1 \cdot 25}{2 \cdot 10} + \frac{1 \cdot 1}{2 \cdot 10} = 40 + 1.25 + 0.05 = 41.3 \text{ W}$$

Fourier Analysis Applications

FRS.07

Find the response $i_o(t)$ in the circuit in Figure if the input voltage $v(t)$ has the Fourier series expansion as the following

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} (\cos nt - n \sin nt)$$



Solution:

$$A_0 = 1, \quad A_k = \frac{2(-1)^n}{1+n^2}, \quad B_k = -\frac{2n(-1)^n}{1+n^2}, \quad C_k = \sqrt{A_k^2 + B_k^2} = \frac{2(-1)^n}{\sqrt{1+n^2}}, \quad \varphi_k = -\tan^{-1} \left(\frac{B_k}{A_k} \right) = \tan^{-1} n$$

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{1+n^2}} \cos(nt + \tan^{-1} n) = 1 - 1.41 \cos(t + 45^\circ) + 0.89 \cos(2t + 63.45^\circ) - 0.63 \cos(3t + 71.56^\circ) + \dots$$

$$\mathbf{Z}_n = 4 + j\omega_n 2 \times 4 = 4 + \frac{j\omega_n 8}{4 + j\omega_n 2} = \frac{8 + j\omega_n 8}{2 + j\omega_n}$$

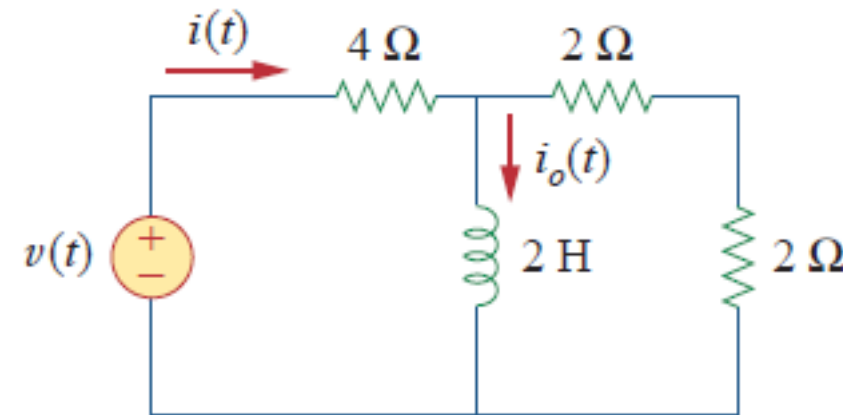
Fourier Analysis Applications

$$\mathbf{Z} = \frac{8 + j\omega_n 8}{2 + j\omega_n} \rightarrow \mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{2 + j\omega_n}{8 + j\omega_n 8} \mathbf{V}$$

$$\mathbf{I}_0 = \frac{4}{4 + j\omega_n 2} \mathbf{I} = \frac{\mathbf{V}}{4 + j\omega_n 4}$$

$$\omega_n = 0 \rightarrow V = 1 \rightarrow \mathbf{I}_0 = \frac{\mathbf{V}}{4} = \frac{1}{4}$$

$$\omega_n = n \rightarrow \mathbf{V} = \frac{2(-1)^n}{\sqrt{1 + n^2}} e^{j \tan^{-1} n} \rightarrow \mathbf{I}_0 = \frac{\mathbf{V}}{4 + j4n} = \dots = \frac{(-1)^n}{2(1 + n^2)}$$



$$i_0(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2(1 + n^2)} \cos nt \text{ A}$$

Fourier Analysis Applications

FRS.08

Find an estimate for the RMS value of the voltage given by Fourier series expansion.

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} (\cos nt - n \sin nt)$$

Solution:

$$A_0 = 1, \quad A_k = \frac{2(-1)^k}{1+k^2}, \quad B_k = \frac{2k(-1)^k}{1+k^2}, \quad C_k = \sqrt{A_k^2 + B_k^2} = \frac{2(-1)^k}{\sqrt{1+k^2}}, \quad \varphi_k = \tan^{-1} \frac{B_k}{A_k} = \tan^{-1} k$$

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{1+n^2}} \cos(nt + \tan^{-1} n) = 1 - 1.41 \cos(t + 45^\circ) + 0.89 \cos(2t + 63.45^\circ) \\ - 0.63 \cos(3t + 71.56^\circ) - 0.48 \cos(4t + 78.7^\circ) + \dots$$

$$V_{rms} = \sqrt{A_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} A_k^2} \approx \sqrt{1^2 + \frac{1}{2} [(-1.41)^2 + (0.89)^2 + (-0.63)^2 + (-0.48)^2 + \dots]} = \sqrt{2.72} = 1.65 \text{ V}$$

NOTE: Estimate! Original func $\rightarrow v(t) = \frac{\pi e^t}{\sinh \pi}$, $-\pi < t < \pi$, $v(t) (= v(t+T)) \rightarrow V_{RMS} = 1.776 \text{ V}$

