DR. GYURCSEK ISTVÁN

Multi-Wave Signals and Circuits Exercises

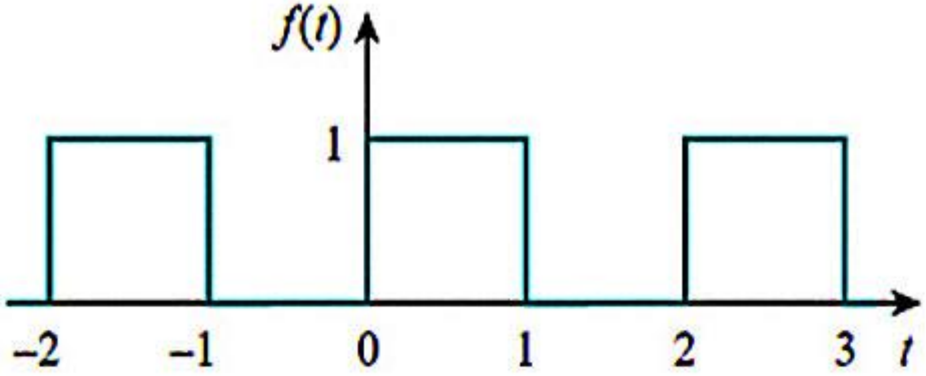
Sources and additional materials (recommended)

- *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*

Fourier Analysis

FRS.01

Determine the Fourier series of the waveform.
Obtain the amplitude and phase spectra.



Solution:

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \qquad \omega_0 = \frac{2\pi}{T} = \pi$$

$$A_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}\left(\int_0^1 1dt + \int_1^2 0dt\right) = \frac{1}{2}$$

Fourier Analysis

$$A_k = \frac{2}{T}\int_0^T f(t)\cos k\omega_0 t\,dt = \frac{2}{2}\left(\int_0^1 1\cos k\pi t\,dt + \int_1^2 0\cos k\pi t\,dt\right) = \frac{1}{k\pi}\sin k\pi = 0$$

$$B_k = \frac{2}{T}\int_0^T f(t)\sin k\omega_0 t\,dt = \frac{2}{2}\left(\int_0^1 1\sin k\pi t\,dt + \int_1^2 0\sin k\pi t\,dt\right) = \left.\frac{-\cos k\pi t}{k\pi}\right|_0^1 = \frac{1}{k\pi}(-\cos k\pi + 1)$$

$$B_k = \begin{cases} {}^{2}\!/_{k\pi} & k = \ odd\ (PN) \\ 0 & k = even\ (PS) \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \frac{2}{7\pi}\sin 7\pi t + \cdots$$

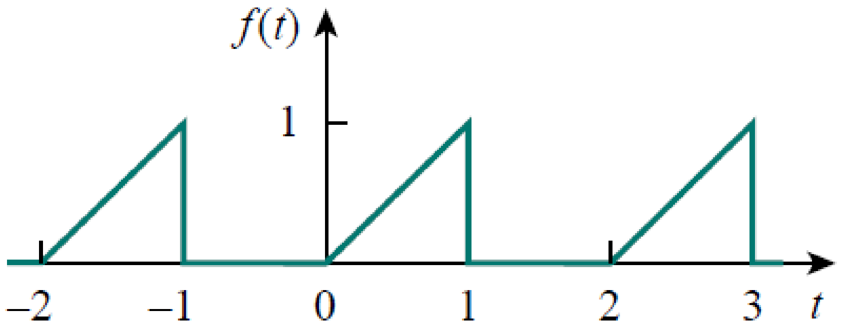
$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{n}\sin n\pi t\,, \qquad n = 2i - 1$$

To obtain spectra: No A_k → C_k = B_k and φ_k = 0.

Fourier Analysis

FRS.02 – Obtain the Fourier series for the periodic function in Figure and plot the amplitude and phase spectra.

Solution: $f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases} \qquad \omega_0 = \frac{2\pi}{T} = \pi$



$$A_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}\left(\int_0^1 t\,dt + \int_1^2 0dt\right) = \frac{1}{2}\left.\frac{t^2}{2}\right|_0^1 = \frac{1}{4}$$

$$A_k = \frac{2}{T}\int_0^T f(t)\,cos\,k\omega_0 t\,dt = \frac{2}{2}\left(\int_0^1 \mathrm{t}\,cos\,k\pi t\,dt + 0\right) = \left[\frac{1}{k^2\pi^2}\cos k\pi t + \frac{1}{k\pi}\sin k\pi t\right]\Bigg|_0^1 = \frac{1}{k^2\pi^2}(\cos k\pi - 1) + 0 = \frac{(-1)^k - 1}{k^2\pi^2}$$

$$B_k = \frac{2}{T}\int_0^T f(t)\,sin\,k\omega_0 t\,dt = \frac{2}{2}\left(\int_0^1 \mathrm{t}\,sin\,k\pi t\,dt + \int_1^2 0\,sin\,k\pi t\,dt\right) = \left[\frac{1}{k^2\pi^2}\sin k\pi t - \frac{t}{k\pi}\cos k\pi t\right]\Bigg|_0^1 = 0 - \frac{\cos k\pi}{k\pi} = \frac{(-1)^{k+1}}{k\pi}$$

$$f(t) = \frac{1}{4} + \sum_{i=1}^{\infty}\left[\frac{(-1)^k - 1}{(k\pi)^2}\cos k\pi t + \frac{(-1)^{k+1}}{k\pi}\,sin\,k\pi t\right]$$

Fourier Analysis

$$f(t) = \frac{1}{4} + \sum_{i=1}^{\infty} \left[\frac{(-1)^k - 1}{(k\pi)^2} \cos k\pi t + \frac{(-1)^{k+1}}{k\pi} sin\, k\pi t \right]$$

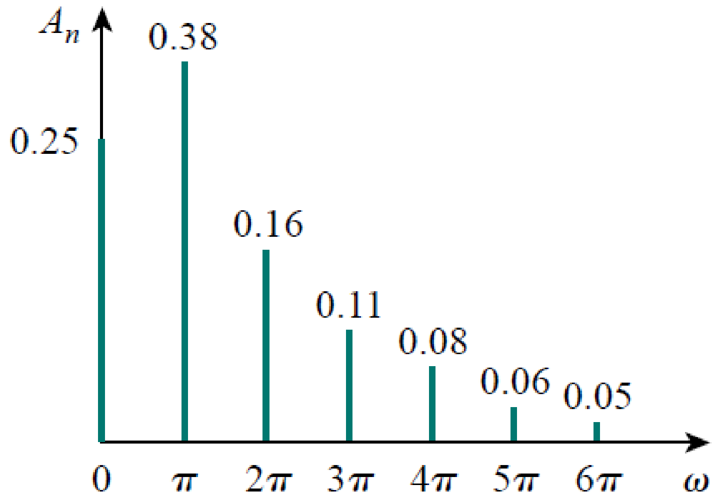
$$A_k = 0, B_k = -\frac{1}{k\pi} \leftarrow k = 2, 4, \ldots$$

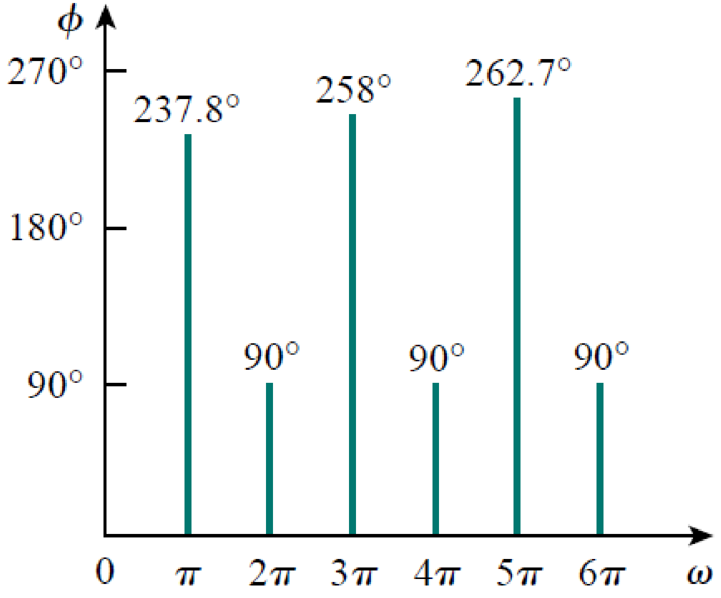
$$\boldsymbol{C}_k = A_k - jB_k = 0 + j\frac{1}{k\pi} = \frac{1}{k\pi} e^{j90°} \leftarrow k = 2, 4, \ldots$$

$$A_k = -\frac{2}{(k\pi)^2}, B_k = \frac{1}{k\pi} \leftarrow k = 1, 3, \ldots$$

$$\boldsymbol{C}_k = A_k - jB_k = -\frac{2}{(k\pi)^2} - j\frac{1}{k\pi} = C_k e^{j\varphi_k} \leftarrow k = 1, 3, \ldots$$

$$\left. \begin{array}{l} C_k = \sqrt{A_k^2 + B_k^2} = \sqrt{\dfrac{4}{(k\pi)^4} + \dfrac{1}{(k\pi)^2}} = \dfrac{\sqrt{4 + (k\pi)^2}}{(k\pi)^2} \\ \varphi_k = 180° + tan^{-1} \dfrac{k\pi}{2} \end{array} \right\} \leftarrow k = 1, 3, \ldots$$





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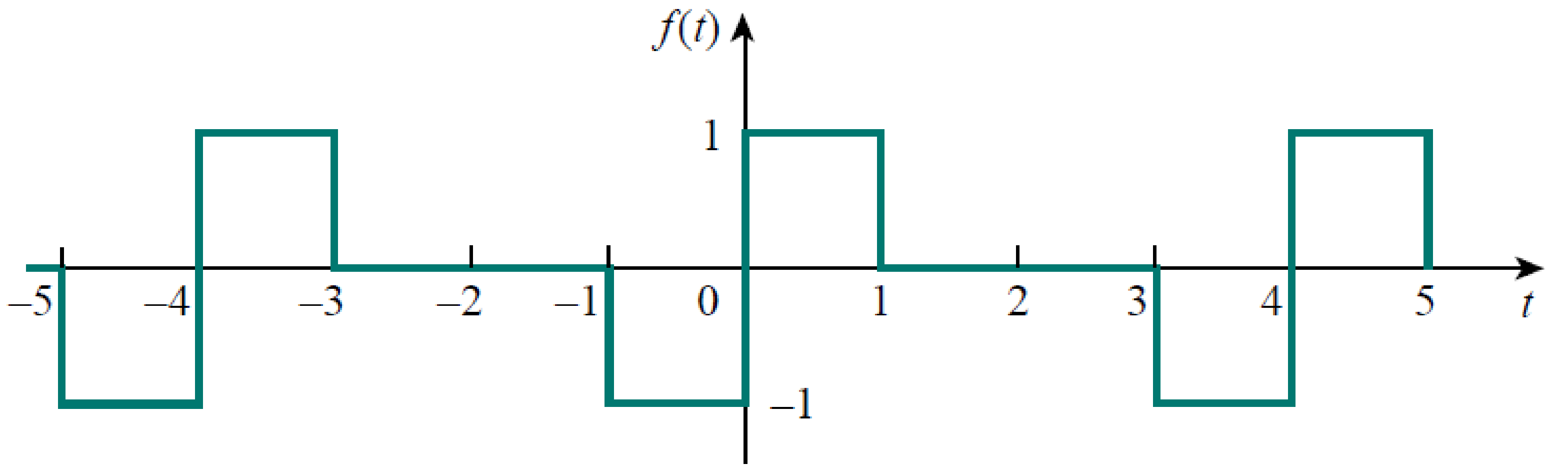
Fourier Analysis

FRS.03 – Find the Fourier series expansion of $f(t)$ given in Figure.

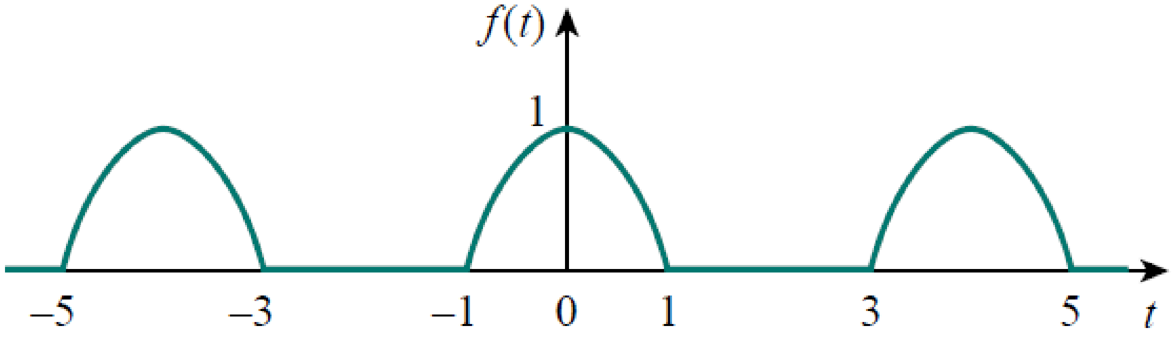


Result: $$f(t) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \left(1 - \cos \frac{k\pi}{2} \right) \sin \frac{k\pi}{2} t$$

Fourier Analysis

FRS.04

Determine the Fourier series for the half-wave rectified cosine function shown in Figure..



Result: $$f(t) = \frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi}{2}t - \frac{2}{\pi}\sum_{k\_1}^{\infty}\frac{(-1)^k}{4k^2-1}\cos k\pi t$$

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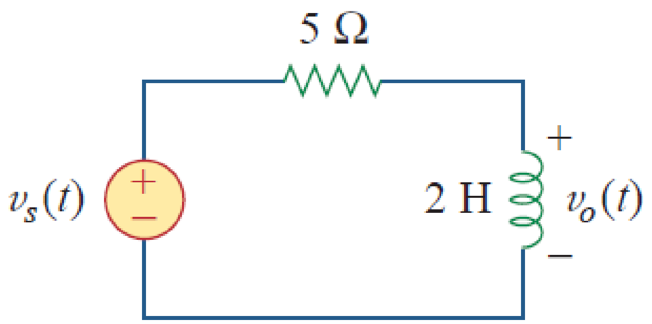
Applications

FRS.05

Find the $v_0(t)$ response voltage across the inductor in the circuit if the source voltage is

$$v_S(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n} sin\, n\pi t\,, \qquad n = 2k - 1$$



Solution:

$$\boldsymbol{V}_{0n} = \boldsymbol{V}_{Sn}\frac{j\omega_n L}{R + j\omega_n L} = \boldsymbol{V}_{Sn}\frac{j2n\pi}{5 + j2n\pi}$$

$$\omega_n = n \cdot \omega_1 = n \cdot \pi$$

n = 0

$$V_{S0} = \frac{1}{2} \rightarrow V_{00} = 0$$

Applications

n-th harmonic

$$v_S(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n} sin\, n\pi t\,, \qquad n = 2k-1$$

$$v_S(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n} cos(n\pi t - 90°)\,, \qquad n = 2k-1$$

$$\boldsymbol{V}_{Sn} = \frac{2}{n\pi} e^{-j90°} = \frac{-j2}{n\pi} \rightarrow \boldsymbol{V}_{\boldsymbol{0}n} = \frac{-j2}{n\pi}\frac{j2n\pi}{5 + j2n\pi} = \frac{4e^{-jtan^{-1}\frac{2n\pi}{5}}}{\sqrt{25 + 4n^2\pi^2}}$$

$$v_0(t) = 0 + \sum_{k=1}^{\infty}\frac{4}{\sqrt{25 + 4n^2\pi^2}} cos\left(n\pi t - tan^{-1}\frac{2n\pi}{5}\right), \qquad n = 2k-1$$

$$v_0(t) = 0.50\, cos(\pi t - 51.5°) + 0.2\, cos(3\pi t - 75°) + 0.13\, cos(5\pi t - 81°) + \cdots$$

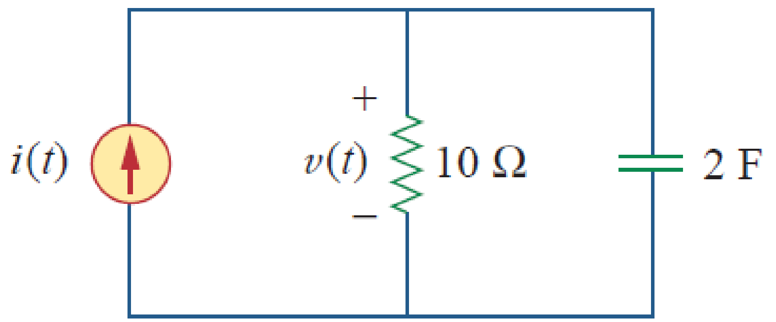
Applications

FRS.06

Determine the average power supplied to the circuit if $i(t) = 2 + 10\,cos(t + 10°) + 6\,cos(3t + 45°)\ A$

Solution: $\boldsymbol{Z} = 10 \times \dfrac{1}{j2\omega} = \dfrac{10}{1 + j20\omega}$

$$\boldsymbol{V} = \boldsymbol{I}\,\boldsymbol{Z} = \frac{10\,\boldsymbol{I}}{1 + j20\omega} = \frac{10\,\boldsymbol{I}}{\sqrt{1 + 400\omega^2}\ e^{j tan^{-1} 20\omega}}$$



$$\omega = 0 \rightarrow \boldsymbol{I}_0 = 2\ A \rightarrow \boldsymbol{V} = 2 \cdot 10 = 20\ V$$

$$\omega = 1 \rightarrow \boldsymbol{I}_1 = 10\ e^{j10°} \rightarrow \boldsymbol{V} = \frac{10 \cdot 10\ e^{j10°}}{\sqrt{1 + 400}\ e^{j tan^{-1} 20}} = 5\ e^{-j77.14°}\ V$$

$$\omega = 3 \rightarrow \boldsymbol{I}_3 = 6\ e^{j45°} \rightarrow \boldsymbol{V} = \frac{10 \cdot 6\ e^{j45°}}{\sqrt{1 + 3600}\ e^{j tan^{-1} 60}} = 1\ e^{-j44.05°}\ V$$

$$v(t) = 20 + 5\,cos(t - 77.14°) + 1\,cos(3t - 44.05°)\ V$$

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Applications

$$i(t) = 2 + 10\,cos(t + 10°) + 6\,cos(3t + 45°)\ \ A$$

$$v(t) = 20 + 5\,cos(t - 77.14°) + 1\,cos(3t - 44.05°)\ \ V$$

Average power

$$P = V_0 I_0 + \frac{1}{2}\sum_{n=1}^{\infty} V_n I_n\,cos(\varphi_{Un} - \varphi_{In})\ \ = 20 \cdot 2 + \frac{5 \cdot 10}{2} cos(-77.14° - 10°) + \frac{1 \cdot 6}{2}\ cos(-44.05° - 45°)$$

$$= 40 + 1.247 + 0.05 = 41.3\,W$$

Alternatively

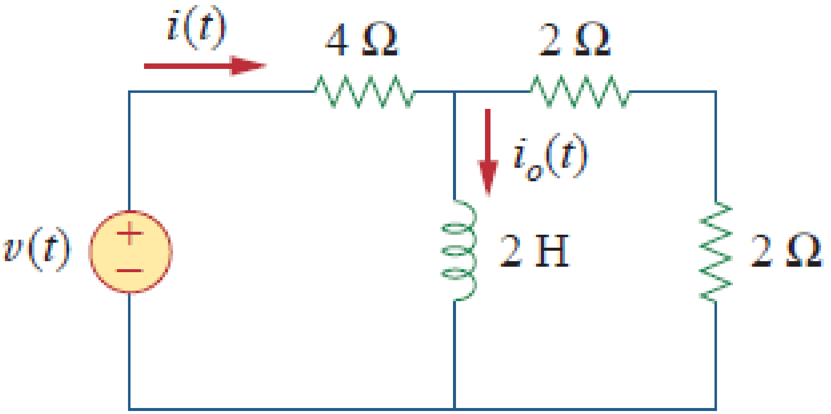
$$P = \frac{V_0^2}{R} + \frac{1}{2}\sum_i \frac{V_i^2}{R} = \frac{400}{10} + \frac{1}{2}\frac{25}{10} + \frac{1}{2}\frac{1}{10} = 40 + 1.25 + 0.05 = 41.3\,W$$

Applications

FRS.07

Find the response $i_o(t)$ in the circuit in Figure if the input voltage v(t) has the Fourier series expansion as the following

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2}(\cos nt - n \sin nt)$$



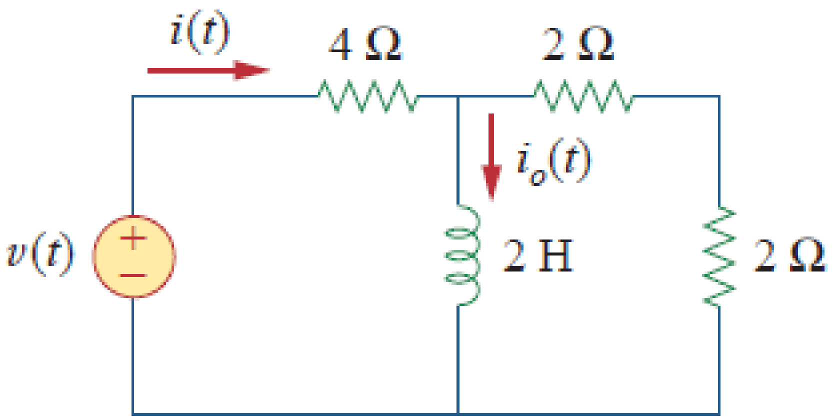
Solution:

$$A_0 = 1, \quad A_k = \frac{2(-1)^n}{1+n^2}, \quad B_k = -\frac{2n(-1)^n}{1+n^2}, \quad C_k = \sqrt{A_k^2 + B_k^2} = \frac{2(-1)^n}{\sqrt{1+n^2}}, \quad \varphi_k = -tan^{-1}\left(\frac{B_k}{A_k}\right) = tan^{-1} n$$

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{1+n^2}} \cos(nt + tan^{-1} n) = 1 - 1.41 \cos(t + 45°) + 0.89 \cos(2t + 63.45°) - 0.63 \cos(3t + 71.56°) + \cdots$$

$$\boldsymbol{Z}_n = 4 + j\omega_n 2 \times 4 = 4 + \frac{j\omega_n 8}{4 + j\omega_n 2} = \frac{8 + j\omega_n 8}{2 + j\omega_n}$$

Applications



$$\boldsymbol{Z} = \frac{8 + j\omega_n 8}{2 + j\omega_n} \rightarrow \boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}} = \frac{2 + j\omega_n}{8 + j\omega_n 8}\boldsymbol{V}$$

$$\boldsymbol{I}_0 = \frac{4}{4 + j\omega_n 2}\boldsymbol{I} = \frac{\boldsymbol{V}}{4 + j\omega_n 4}$$

$$\omega_n = 0 \rightarrow V = 1 \rightarrow \boldsymbol{I}_0 = \frac{\boldsymbol{V}}{4} = \frac{1}{4}$$

$$\omega_n = n \rightarrow \boldsymbol{V} = \frac{2(-1)^n}{\sqrt{1 + n^2}} e^{j tan^{-1} n} \rightarrow \boldsymbol{I}_0 = \frac{\boldsymbol{V}}{4 + j4n} = \cdots = \frac{(-1)^n}{2(1 + n^2)}$$

$$i_0(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2(1 + n^2)} \cos nt \ \ A$$

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Applications

FRS.08

Find an estimate for the RMS value of the voltage given by Fourier series expansion.

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2}(\cos nt - n \sin nt)$$

Solution:

$$A_0 = 1, \qquad A_k = \frac{2(-1)^n}{1+n^2}, \qquad B_k = \frac{2n(-1)^n}{1+n^2}, \qquad C_k = \sqrt{A_k^2 + B_k^2} = \frac{2(-1)^n}{\sqrt{1+n^2}}, \qquad \varphi_k = tan^{-1}\frac{B_k}{A_k} = tan^{-1} n$$

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{1+n^2}} \cos(nt + tan^{-1} n) = 1 - 1.41\cos(t + 45°) + 0.89\cos(2t + 63.45°)$$

$$-0.63\cos(3t + 71.56°) - 0.48\cos(4t + 78.7°) + \cdots$$

$$V_{rms} = \sqrt{A_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} C_k^2} \approx \sqrt{1^2 + \frac{1}{2}[(-1.41)^2 + (0.89)^2 + (-0.63)^2 + (-0.48)^2 + \cdots]} = \sqrt{2.72} = 1.65\ V$$

$$NOTE{:}\ Estimate!\, Original\ func \rightarrow v(t) = \frac{\pi e^t}{\sinh \pi}, -\pi < t < \pi\ , v(t(= v(t+T) \rightarrow V_{RMS} = 1.776\ V$$

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Questions