DR. GYURCSEK ISTVÁN

Exercises with Laplace Transform

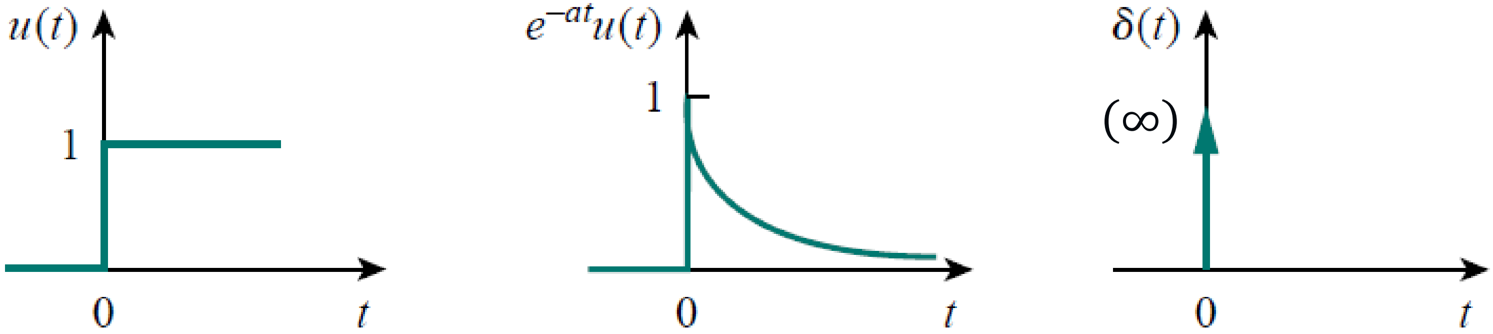
Sources and additional materials (recommended)

- *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*

Laplace Transform

LPT.01 – Determine the Laplace transform of each of the following functions:

$(\mathbf{a})\ \mathrm{u(t)}, (\mathbf{b})\ e^{-at}, a \geq 0\ (\boldsymbol{c})\ \delta(t)$



Solution

$$(\mathbf{a})\ \mathcal{L}\{u(t)\} = \int_0^\infty 1\, e^{-st}\, dt = -\frac{1}{s} e^{-st}\bigg|_0^\infty = -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s}$$

$$(\mathbf{b})\ \mathcal{L}\{e^{-at}u(t)\} = \int_0^\infty e^{-at}e^{-st}\, dt = -\frac{1}{s+a} e^{-(s+a)t}\bigg|_0^\infty = \frac{1}{s+a}$$

$$(\mathbf{c})\ \mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t)\, e^{-st}\, dt = e^{-0} = 1$$

$$unit\ pulse \rightarrow \delta(t) = \frac{du(t)}{dt} \rightarrow \int_{-0}^{+0} \delta(t)\, dt = 1$$

$$APPLICATION:\ sampling\ or\ shifting\ property \rightarrow \int_{-0}^{+0} f(t)\cdot\delta(t)\, dt = f(0)$$

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Laplace Transform

LPT.02 – Determine the Laplace transform of $f(t) = u(t) \cdot \sin \omega t$

Solution

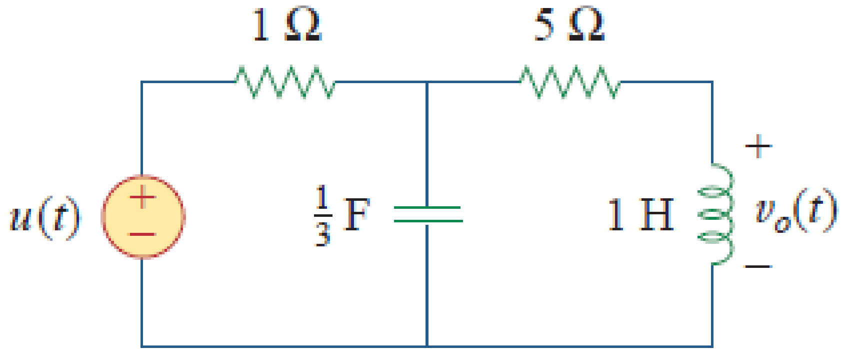
$$F(s) = \mathcal{L}\{\sin \omega t\} = \int_0^{\infty} (\sin \omega t)\, e^{-st}\, dt = \int_0^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st}\, dt$$

$$= \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) = \frac{\omega}{s^2 + \omega^2}$$

Laplace Transform

Steps in applying the Laplace transform :

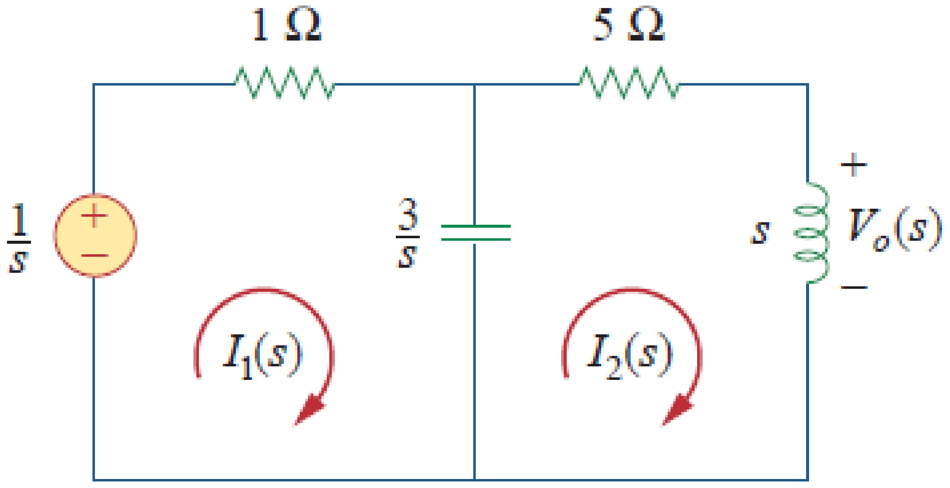
- 1. Transform the circuit from the time domain to the *s* domain
- 2. Solve the circuit using any circuit analysis technique (nodal, mesh analysis, source transformation, superposition, …)
- 3. Take the inverse transform of the solution and thus obtain the solution in the time domain.



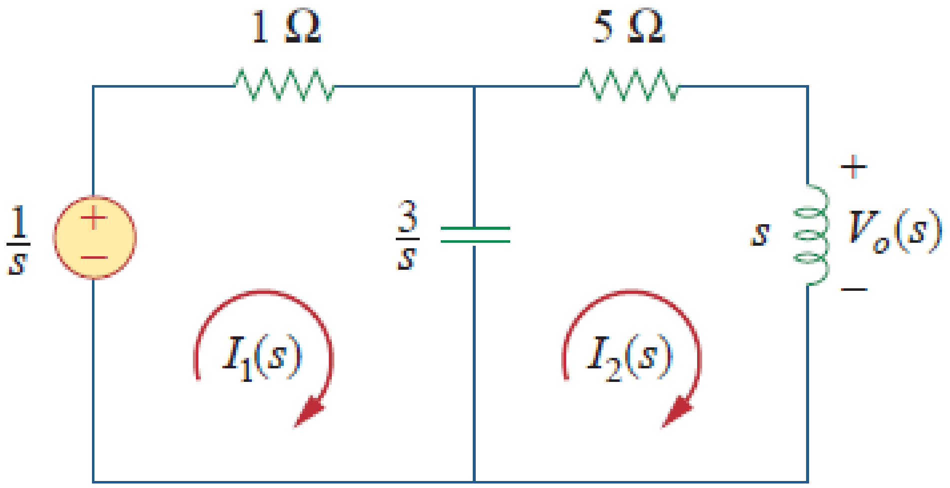
LPT.03 – Find $v_o(t)$ in the circuit in Figure, assuming zero initial conditions.

Solution

$$\left.\begin{matrix} u(t) \\ 1\ H \\ \frac{1}{3}\ F \end{matrix}\right\} \rightarrow \left\{\begin{matrix} \frac{1}{s} \\ sL = s \\ \frac{1}{sC} = \frac{3}{s} \end{matrix}\right.$$



Laplace Transform



$$(1)\ \frac{1}{s} = \left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2$$

$$(2)\ 0 = -\frac{3}{s} I_1 + \left(s + 5 + \frac{3}{s}\right) I_2 \to I_1 = \frac{1}{3}(s^2 + 5s + 3) I_2$$

$$(2) \to (1);\ \frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3) I_2 - \frac{3}{s} I_2$$

$$3 = (s^3 + 8s^2 + 18s) I_2 \to I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_0(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{(s+4)^2 + \sqrt{2}^2}$$

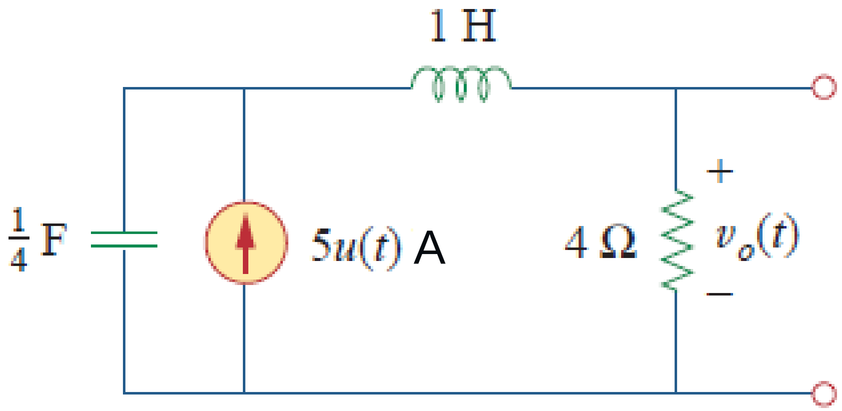
$$\mathcal{L}^{-1}\left\{\frac{\omega_d}{(s+a)^2 + \omega_d^2}\right\} = e^{-at} \cdot \sin(\omega_d t)$$

$$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = \frac{3}{\sqrt{2}} e^{-4t} \sin\left(\sqrt{2}\, t\right) u(t)\ V, t \geq 0$$

Applications

LPT.04

Determine $v_o(t)$ in the circuit, assuming zero initial conditions.



Solution

$current\ division \rightarrow$

$$\left.\begin{matrix} 5 \cdot u(t) \\ \\ 1\ H \\ \\ \dfrac{1}{4}\ F \end{matrix}\right\} \rightarrow \begin{cases} \dfrac{5}{s} \\ sL = s \\ \dfrac{1}{sC} = \dfrac{4}{s} \end{cases}$$

$$I_0(s) = \frac{5}{s} \cdot \frac{4/s}{4/s + 4 + s} = \frac{20}{s} \cdot \frac{1}{4 + 4s + s^2} = 20 \cdot \frac{1}{s(s+2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+a)^2}\right\} = \frac{1}{a^2}\left(1 - e^{-at} - ate^{-at}\right)$$

$$i_0(t) = \mathcal{L}^{-1}\{I_0(s)\} = 20 \cdot \frac{1}{4} \cdot \left(1 - e^{-2t} - 2te^{-2t}\right)$$

$$v_0(t) = 4 \cdot i_0(t) = 20 \cdot \left(1 - e^{-2t} - 2te^{-2t}\right)u(t)\ V$$

Applications

LPT.05 – Determine the $v_0(t)$ voltage for t > 0 across the capacitor in a series RC circuit if R = 100 Ω , C = 100 μF. The circuit is energy free at t = 0 and source voltage is $v_S(t) = u(t) \cdot 20 \cdot \sin(100t + 45°)\ V$

Solution 1 (by Laplace transform)

$$\mathcal{L}[\sin(at+b)] = \frac{s\sin b + a\cos b}{s^2+a^2} \rightarrow V_s(s) = 20\frac{s\frac{\sqrt{2}}{2} + 100\frac{\sqrt{2}}{2}}{s^2+100^2} = 10\sqrt{2}\frac{s+100}{s^2+100^2}$$

$$H(s) = \frac{\frac{1}{sC}}{R+\frac{1}{sC}} = \frac{1}{sRC+1} = \frac{1}{s\,10^{-2}+1} = \frac{100}{s+100}$$

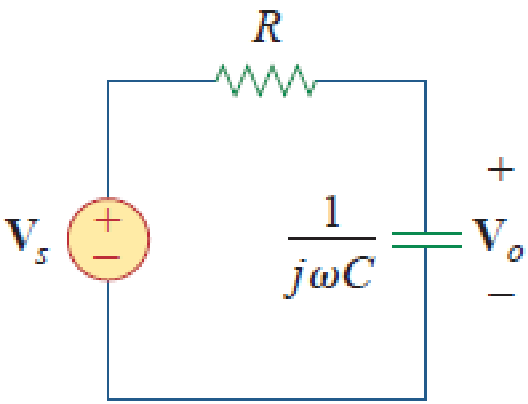
$$V_0(s) = V_S(s)\cdot H(s) = 10\sqrt{2}\frac{s+100}{s^2+100^2}\cdot\frac{100}{s+100} = 10\sqrt{2}\frac{100}{s^2+100^2}$$

$$\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin(at) \rightarrow v_0(t) = \mathcal{L}^{-1}\left[V_C(s)\right] = u(t)\cdot 10\sqrt{2}\sin 100t$$

($V_{0\,rms}$ = 10 V)

Applications

Solution 2 (in phasor domain)



$$v_S(t) = 20 \cdot \sin(100t + 45°)\ V \rightarrow \boldsymbol{V}_S = 20 \cdot e^{j45°}$$

$$\boldsymbol{H}(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1+j} = \frac{1}{\sqrt{2}} e^{-j45°}$$

$$\boldsymbol{V}_0 = \boldsymbol{V}_S \cdot \boldsymbol{H}(\omega) = 20 \cdot e^{j45°} \cdot \frac{1}{\sqrt{2}} e^{-j45°} = 10\sqrt{2}$$

$$v_0(t) = 10\sqrt{2} \sin 100t\,, \qquad t > 0$$

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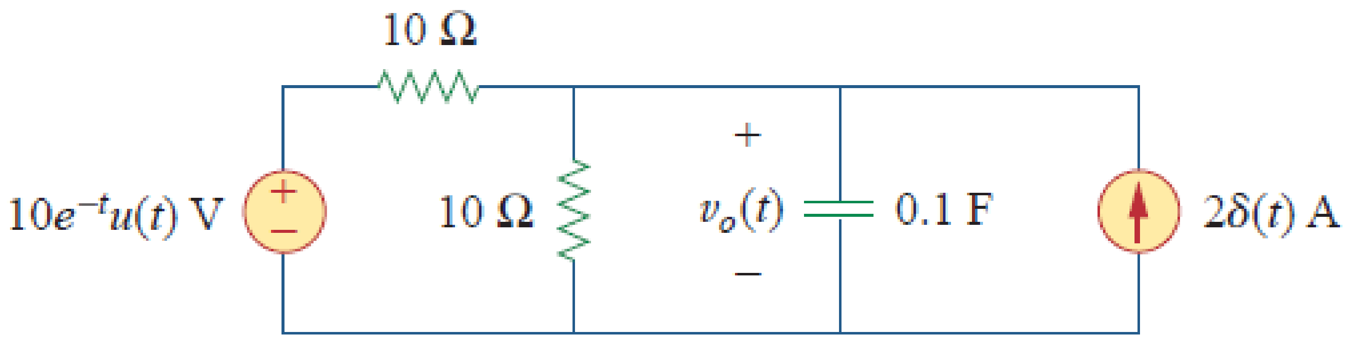
Applications

LPT.06

Obtain $v_o(t)$ in the circuit, assuming $v_o(0) = 5$ V.



Solution $\mathcal{L}\{e^{-at}u(t)\} = \dfrac{1}{s+a} \leftarrow \mathbf{LPL.01}$

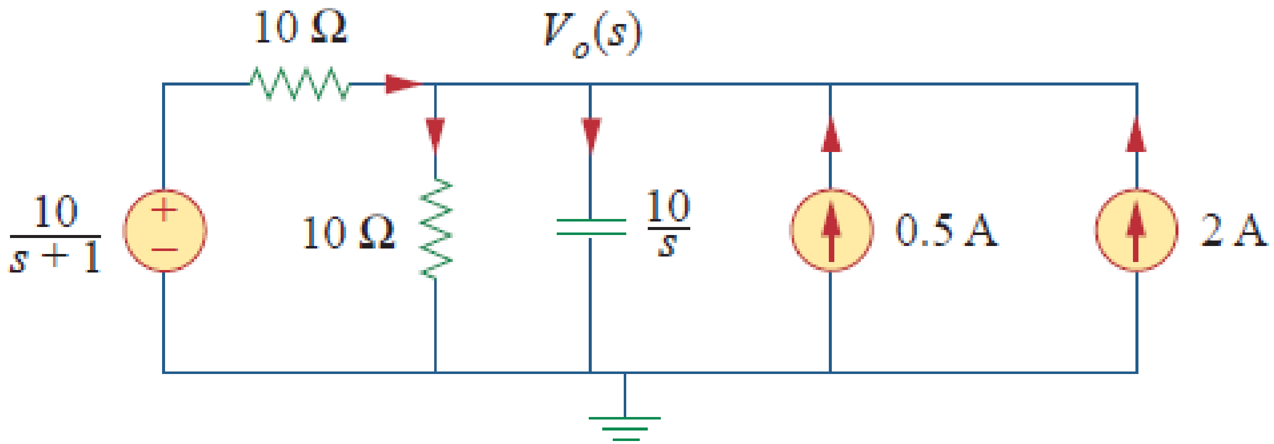
$$init.cond. \rightarrow C \cdot v_0(0) = 0.1 \cdot 5 = 0.5\ A$$

$$\frac{10/(s+1) - V_0}{10} + 2 + 0.5 = \frac{V_0}{10} + \frac{V_0}{10/s}$$

$$\frac{1}{s+1} + 2.5 = \frac{2V_0}{10} + \frac{sV_0}{10} = \frac{1}{10}V_0(s+2)$$

$$\frac{10}{s+1} + 25 = V_0(s+2)$$

$$V_0 = \frac{25s+35}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$



☜ Partial-faction decomposition

Applications

$$V_0(s) = \frac{25s + 35}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$A = (s+1) \cdot V_0(s)\Big|_{s=-1} = \frac{25s+35}{(s+2)}\Bigg|_{s=-1} = \frac{10}{1} = 10$$

$$B = (s+2) \cdot V_0(s)\Big|_{s=-2} = \frac{25s+35}{(s+1)}\Bigg|_{s=-2} = \frac{-15}{-1} = 15$$

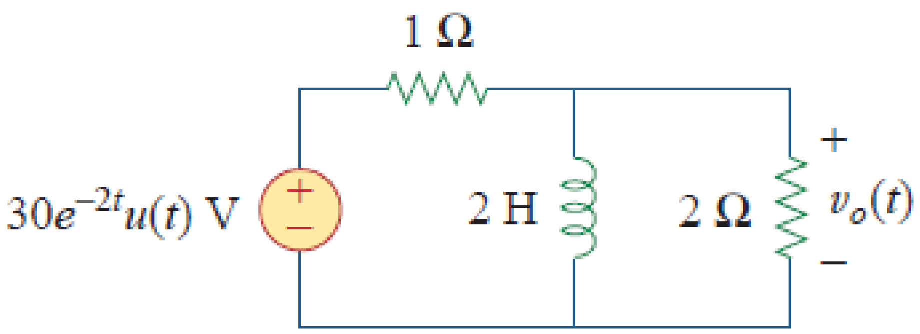
$$V_0(s) = \frac{10}{(s+1)} + \frac{15}{(s+2)}$$

$$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = (10 \cdot e^{-t} + 15 \cdot e^{-2t}) \cdot u(t)\ V$$

Applications

LPT.07 – Find $v_o(0)$. *(Note → since the voltage input is multiplied by u(t), the voltage source is a short for all t < 0 and $i_L(0)=0$.)*

Solution $\mathcal{L}\{e^{-at}u(t)\} = \dfrac{1}{s+a} \leftarrow \mathbf{LPL.01}$ $\quad V(s) = \dfrac{30}{s+2}$



$$V_0(s) = V(s)\cdot\frac{2\times 2s}{2\times 2s+1} = \frac{30}{s+2}\cdot\frac{4s/(2+2s)}{4s/(2+2s)+1}$$

$$= \frac{30}{s+2}\cdot\frac{4s}{4s+2+2s} = \frac{30}{s+2}\cdot\frac{2s}{3s+1} = \frac{20s}{(s+2)\cdot\left(s+\frac{1}{3}\right)} = \frac{A}{(s+2)} + \frac{B}{\left(s+\frac{1}{3}\right)}$$

☜ Partial-faction decomposition

$$A = (s+2)\cdot V_0(s)\Big|_{s=-2} = \frac{20s}{\left(s+\frac{1}{3}\right)}\Bigg|_{s=-2} = 24$$

$$V_0(s) = \frac{24}{(s+2)} - \frac{4}{(s+1/3)}$$

$$B = \left(s+\frac{1}{3}\right)\cdot V_0(s)\Bigg|_{s=-\frac{1}{3}} = \frac{20s}{(s+2)}\Bigg|_{s=-\frac{1}{3}} = -4$$

$$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = \left(24\cdot e^{-2t} - 4\cdot e^{-\frac{t}{3}}\right)\cdot u(t)\ V$$

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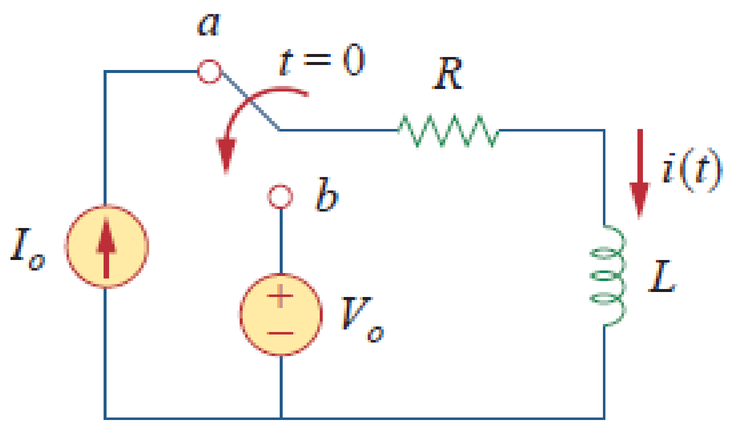
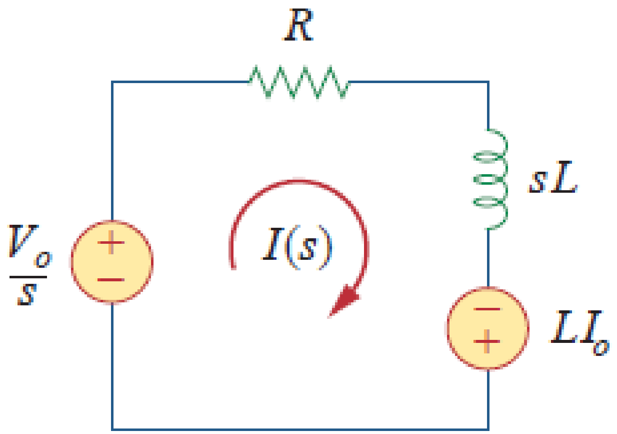
Applications

LPT.08 – The switch moves from position *a* to position *b* at t = 0. Find i(t) for t > 0.





Solution $i(0) = I_0$

$$I(s)(R + sL) - L \cdot I_0 - \frac{V_0}{s} = 0$$

$$I(s) = L \cdot \frac{I_0}{R + sL} + \frac{V_0}{s(R + sL)} = \frac{I_0}{s + {}^{R}/_{L}} + \frac{{}^{V_0}/_{L}}{s\left(s + {}^{R}/_{L}\right)}$$

$$I(s) = \frac{I_0}{s + {}^{R}/_{L}} + \frac{{}^{V_0}/_{R}}{s} - \frac{{}^{V_0}/_{R}}{s + {}^{R}/_{L}} = \frac{I_0 - {}^{V_0}/_{R}}{s + {}^{R}/_{L}} + \frac{{}^{V_0}/_{R}}{s}$$

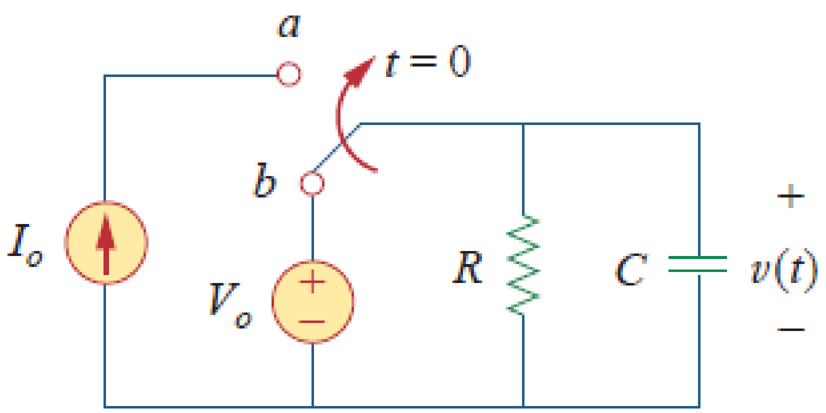
$$i(t) = \left(I_0 - \frac{V_0}{R}\right) e^{-{}^{t}/_{\tau}} + \frac{V_0}{R}\ , t \geq 0$$

$$i(t) = I_0\, e^{-{}^{t}/_{\tau}} + \frac{V_0}{R}\left(1 - e^{-{}^{t}/_{\tau}}\right)\ , t \geq 0$$

Applications

LPT.09 – The switch has been in position b for a long time. It is moved to position a at t = 0. Determine v(t) for t > 0.

Solution $init.cond. \rightarrow i_C(0) = C \cdot v_0(0) = C \cdot V_0$



$$\frac{I_0}{s} + C \cdot V_0 = \frac{V(s)}{R} + s \cdot C \cdot V(s) \rightarrow \frac{I_0}{s} + C \cdot V_0 = V(s) \cdot \left[\frac{1}{R} + s \cdot C\right]$$

$$V(s) = \frac{I_0}{s\left(\frac{1}{R} + s \cdot C\right)} + \frac{C \cdot V_0}{\left(\frac{1}{R} + s \cdot C\right)} = \frac{I_0/C}{s \cdot \left(\frac{1}{R \cdot C} + s\right)} + \frac{V_0}{\frac{1}{R \cdot C} + s}$$

$$V(s) = \frac{I_0 \cdot R}{s} - \frac{I_0 . R}{\frac{1}{R \cdot C} + s} + \frac{V_0}{\frac{1}{R \cdot C} + s} = \frac{I_0 \cdot R}{s} + \frac{V_0 - I_0 \cdot R}{\frac{1}{R \cdot C} + s}$$

$$v(t) = \mathcal{L}^{-1}\{V(s)\} = I_0 \cdot R + (V_0 - I_0 \cdot R)e^{-\frac{t}{\tau}}, \qquad t > 0, \qquad \tau = RC$$

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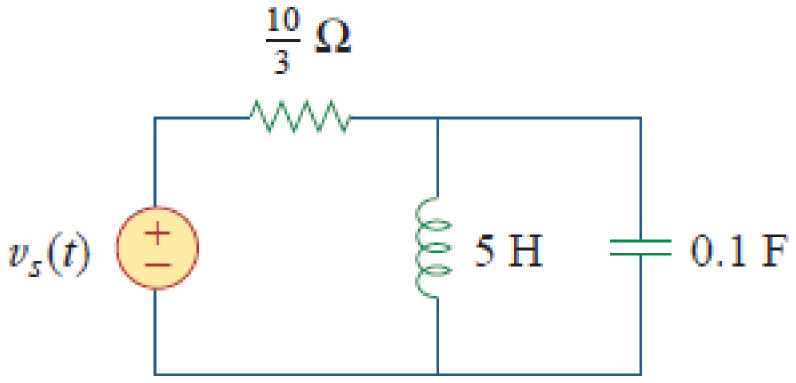
Applications

LPT.10 – Find the voltage across the capacitor assuming that the value of $v_S(t) = 10\,u(t)$ and $i_L(0) = -1$ A , $v_C(0) = +5$ V.



Solution

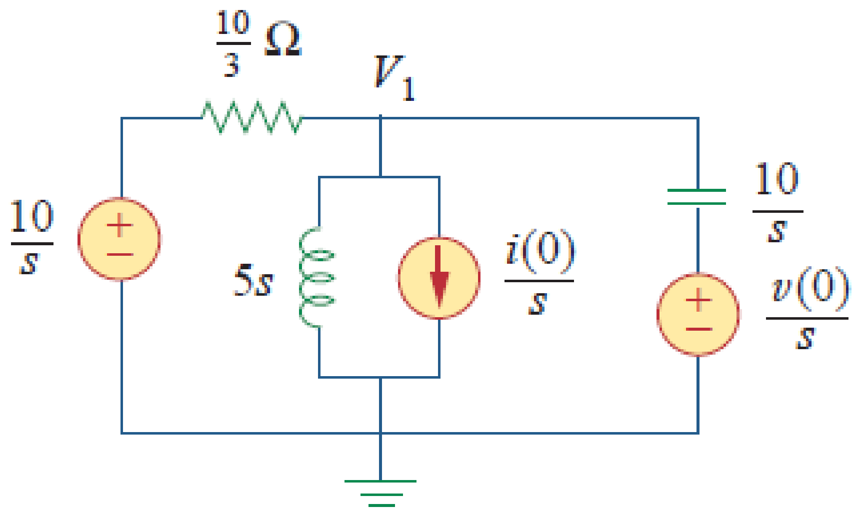
$$\frac{V_1 - {}^{10}/_{s}}{{}^{10}/_{3}} + \frac{V_1}{5s} + \frac{i(0)}{s} + \frac{V_1 - \left[{}^{v(0)}/_{s}\right]}{{}^{10}/_{s}} = 0$$

$$0.1\left(s + 3 + \frac{2}{s}\right)V_1 = \frac{3}{s} + \frac{1}{s} + 0.5$$

$$(s^2 + 3s + 2)V_1 = 40 + 5s$$

$$V_1 = \frac{40 + 5s}{(s+1)(s+2)} = \frac{35}{s+1} - \frac{30}{s+2}$$

$$v_1(t) = (35e^{-t} - 30e^{-2t})u(t)\ V$$



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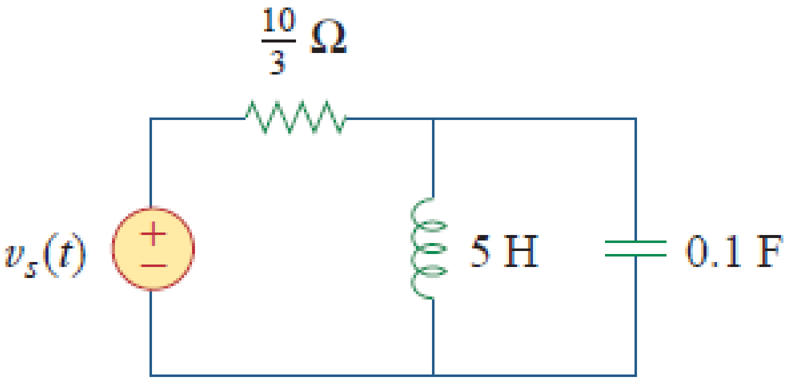
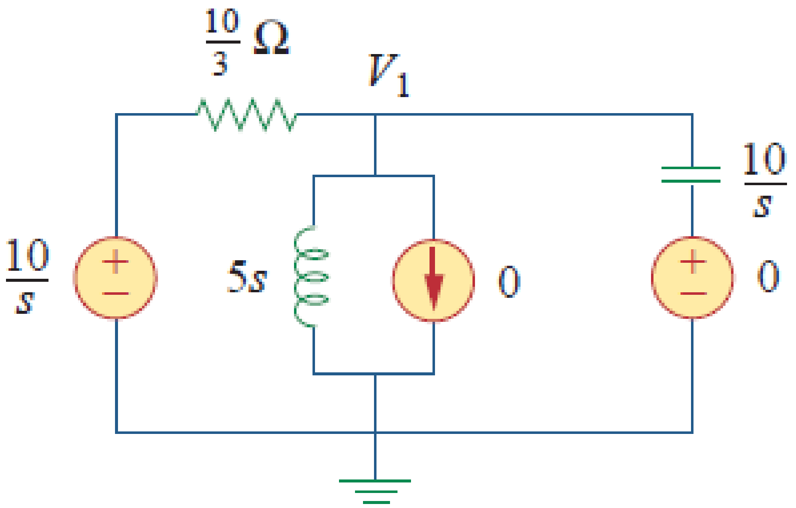
Applications

LPT.11 – Use superposition theorem to find the capacitor voltage. Initial conditions $i_L(0) = -1$ A , $v_C(0) = +5$ V

Solution





$$\frac{V_1 - {}^{10}\!/_{s}}{{}^{10}\!/_{3}} + \frac{V_1 - 0}{5s} + 0 + \frac{V_1 - 0}{{}^{10}\!/_{s}} = 0$$

$$0.1\left(s + 3 + \frac{2}{s}\right)\cdot V_1 = \frac{3}{s}$$

$$(s^2 + 3s + 2)\cdot V_1 = 30$$

$$V_1 = \frac{30}{(s+1)(s+2)} = \frac{30}{s+1} - \frac{30}{s+2}$$

$$v_1(t) = (30\cdot e^{-t} - 30\cdot e^{-2t})\cdot u(t)\ V$$

Applications

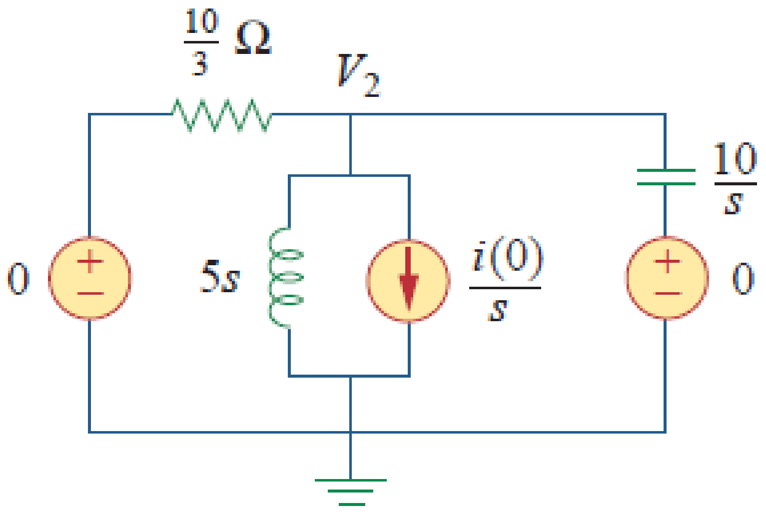
$$\frac{V_2 - 0}{10/_3} + \frac{V_2 - 0}{5s} - \frac{1}{s} + \frac{V_2 - 0}{10/_s} = 0$$

$$0.1\left(s + 3 + \frac{2}{s}\right) \cdot V_2 = \frac{1}{s}$$

$$(s^2 + 3s + 2) \cdot V_2 = 30$$

$$V_2 = \frac{10}{(s+1)(s+2)} = \frac{10}{s+1} - \frac{10}{s+2}$$

$$v_2(t) = (10 \cdot e^{-t} - 10 \cdot e^{-2t}) \cdot u(t)\ V$$



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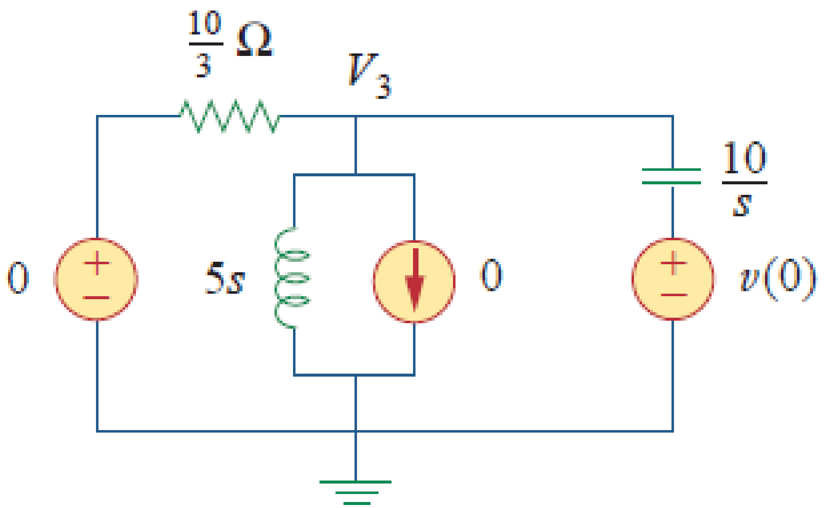
Applications

$$\frac{V_3 - 0}{10/3} + \frac{V_3 - 0}{5s} - 0 + \frac{V_3 - 5/s}{10/s} = 0$$

$$0.1\left(s + 3 + \frac{2}{s}\right) \cdot V_3 = 0.5$$

$$V_3 = \frac{5s}{(s+1)(s+2)} = \frac{-5}{s+1} + \frac{10}{s+2}$$

$$v_3(t) = (-5 \cdot e^{-t} + 10 \cdot e^{-2t}) \cdot u(t)\ V$$



$$v_1(t) = (30 \cdot e^{-t} - 30 \cdot e^{-2t}) \cdot u(t)\ V \qquad v_2(t) = (10 \cdot e^{-t} - 10 \cdot e^{-2t}) \cdot u(t)\ V$$

$$v(t) = v_1(t) + v_2(t) + v_3(t) = \{(30 + 10 - 5) \cdot e^{-t} + (-30 - 10 + 10) \cdot e^{-2t}\} \cdot u(t)$$

$$v(t) = \{35 \cdot e^{-t} - 30 \cdot e^{-2t}\} \cdot u(t)$$ ← *(the same as it was in previous example)*

Questions