



DR. GYURCSEK ISTVÁN

Exercises with Laplace Transform

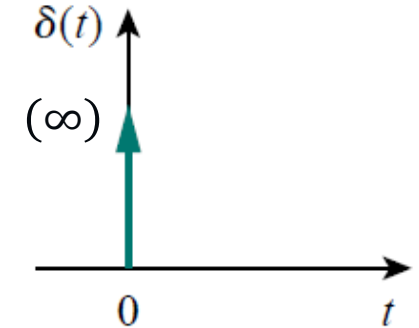
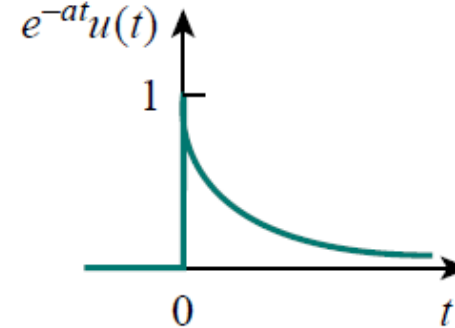
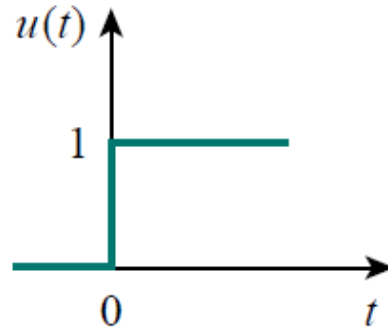
Sources and additional materials (recommended)

- ❑ *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- ❑ *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- ❑ *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- ❑ *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*

Laplace Transform

LPT.01 – Determine the Laplace transform of each of the following functions:

(a) $u(t)$, (b) e^{-at} , $a \geq 0$ (c) $\delta(t)$



Solution (a) $\mathcal{L}\{u(t)\} = \int_0^{\infty} 1 e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = -\frac{1}{s} (0) + \frac{1}{s} (1) = \frac{1}{s}$

(b) $\mathcal{L}\{e^{-at}u(t)\} = \int_0^{\infty} e^{-at} e^{-st} dt = -\frac{1}{s+a} e^{-(s+a)t} \Big|_0^{\infty} = \frac{1}{s+a}$

(c) $\mathcal{L}\{\delta(t)\} = \int_0^{\infty} \delta(t) e^{-st} dt = e^{-0} = 1$ *unit pulse* $\rightarrow \delta(t) = \frac{du(t)}{dt} \rightarrow \int_{-0}^{+0} \delta(t) dt = 1$

APPLICATION: sampling or shifting property $\rightarrow \int_{-0}^{+0} f(t) \cdot \delta(t) dt = f(0)$

LPT.02 – Determine the Laplace transform of $f(t) = u(t) \cdot \sin \omega t$

Solution

$$\begin{aligned} F(s) = \mathcal{L}\{\sin \omega t\} &= \int_0^{\infty} (\sin \omega t) e^{-st} dt = \int_0^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st} dt \\ &= \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) = \frac{\omega}{s^2 + \omega^2} \end{aligned}$$

Laplace Transform

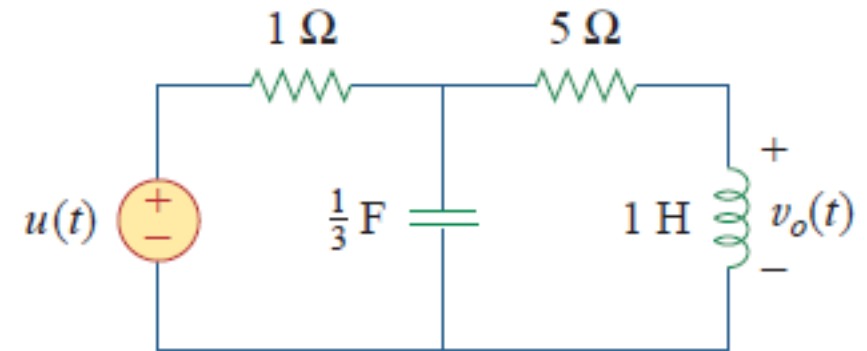
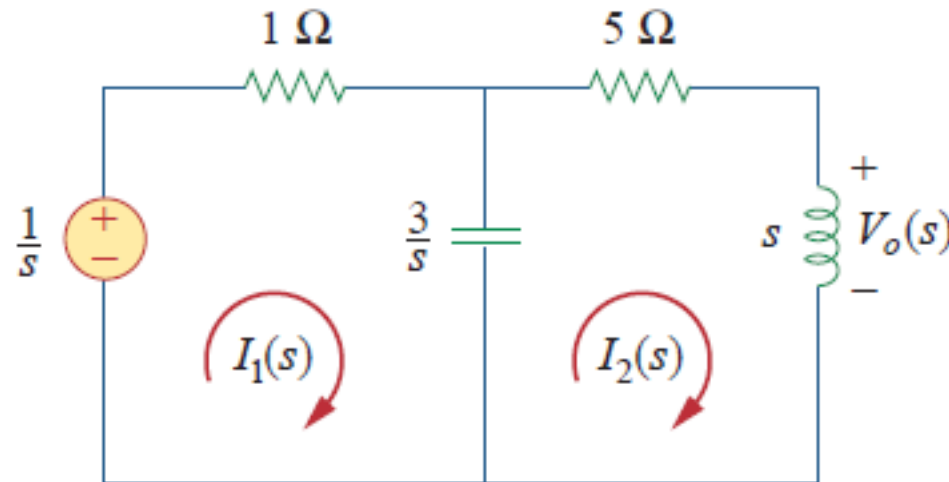
Steps in applying the Laplace transform :

- ❑ 1. Transform the circuit from the time domain to the s domain
- ❑ 2. Solve the circuit using any circuit analysis technique (nodal, mesh analysis, source transformation, superposition, ...)
- ❑ 3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

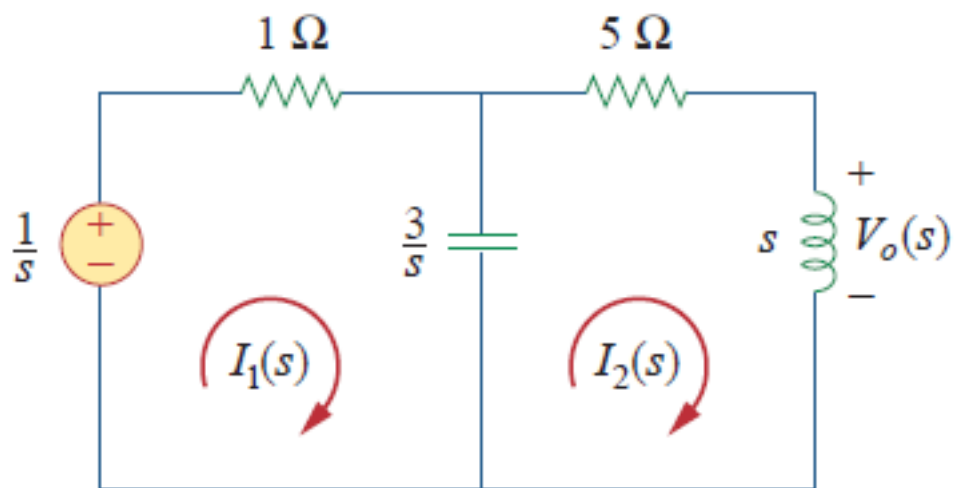
LPT.03 – Find $v_o(t)$ in the circuit in Figure, assuming zero initial conditions.

Solution

$$\left. \begin{array}{l} u(t) \\ 1 \text{ H} \\ \frac{1}{3} \text{ F} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \frac{1}{s} \\ sL = s \\ \frac{1}{sC} = \frac{3}{s} \end{array} \right.$$



Laplace Transform



$$(1) \frac{1}{s} = \left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2$$

$$(2) 0 = -\frac{3}{s} I_1 + \left(s + 5 + \frac{3}{s}\right) I_2 \rightarrow I_1 = \frac{1}{3} (s^2 + 5s + 3) I_2$$

$$(2) \rightarrow (1); \frac{1}{s} = \left(1 + \frac{3}{s}\right) \frac{1}{3} (s^2 + 5s + 3) I_2 - \frac{3}{s} I_2$$

$$3 = (s^3 + 8s^2 + 18s) I_2 \rightarrow I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_0(s) = s I_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{(s + 4)^2 + \sqrt{2}^2}$$

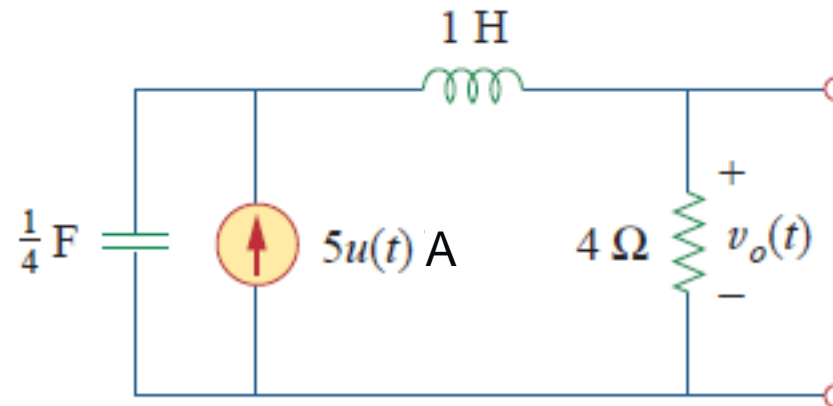
$$\mathcal{L}^{-1} \left\{ \frac{\omega_d}{(s + a)^2 + \omega_d^2} \right\} = e^{-at} \cdot \sin(\omega_d t)$$

$$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = \frac{3}{\sqrt{2}} e^{-4t} \sin(\sqrt{2} t) u(t) \text{ V}, t \geq 0$$

Laplace Transform Applications

LPT.04

Determine $v_o(t)$ in the circuit, assuming zero initial conditions.



Solution

current division \rightarrow

$$\left. \begin{array}{l} 5 \cdot u(t) \\ 1 \text{ H} \\ \frac{1}{4} \text{ F} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \frac{5}{s} \\ sL = s \\ \frac{1}{sC} = \frac{4}{s} \end{array} \right.$$

$$I_0(s) = \frac{5}{s} \cdot \frac{4/s}{4/s + 4 + s} = \frac{20}{s} \cdot \frac{1}{4 + 4s + s^2} = 20 \cdot \frac{1}{s(s + 2)^2}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s + a)^2} \right\} = \frac{1}{a^2} (1 - e^{-at} - ate^{-at})$$

$$i_0(t) = \mathcal{L}^{-1}\{I_0(s)\} = 20 \cdot \frac{1}{4} \cdot (1 - e^{-2t} - 2te^{-2t})$$

$$v_o(t) = 4 \cdot i_0(t) = 20 \cdot (1 - e^{-2t} - 2te^{-2t})u(t) \text{ V}$$

Laplace Transform Applications

LPT.05

Obtain $v_o(t)$ in the circuit, assuming $v_o(0) = 5$ V.

Solution $\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a} \leftarrow \text{LPL. 01}$

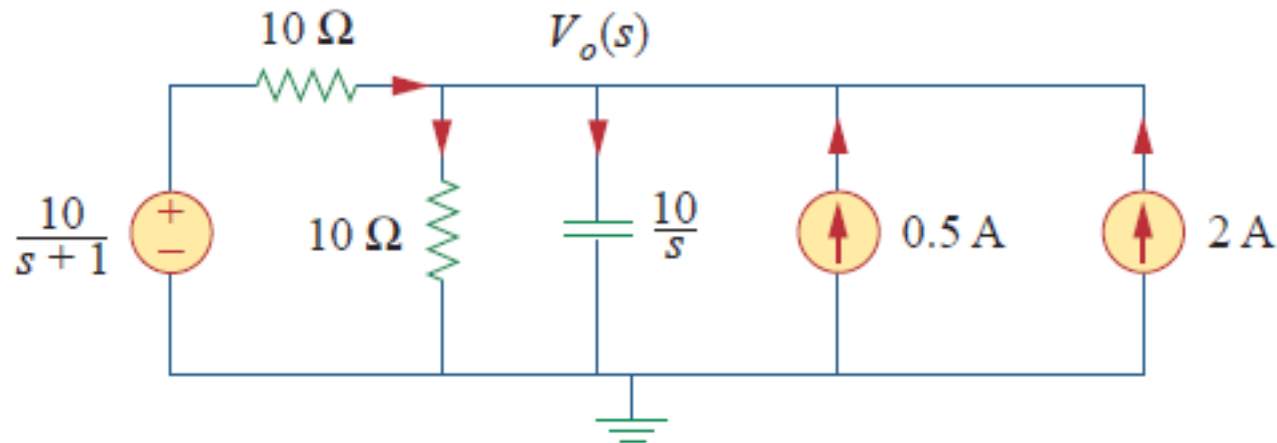
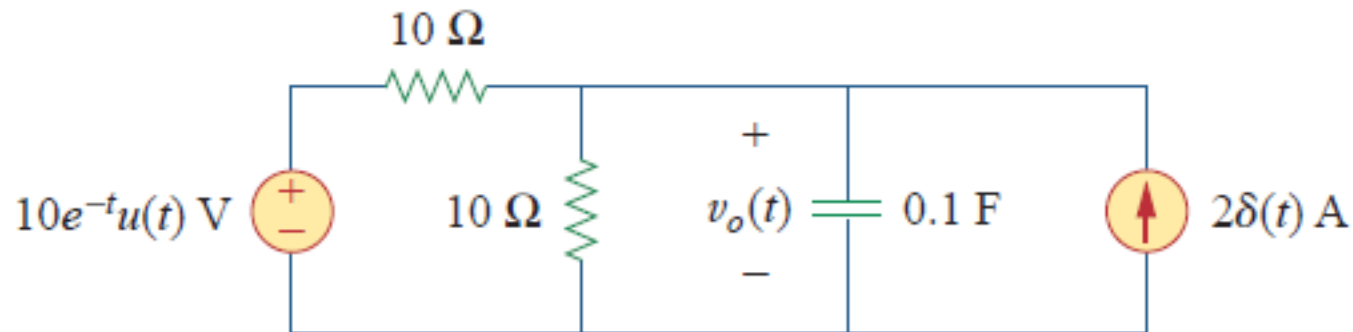
init. cond. $\rightarrow C \cdot v_o(0) = 0.1 \cdot 5 = 0.5$ A

$$\frac{10/(s+1) - V_0}{10} + 2 + 0.5 = \frac{V_0}{10} + \frac{V_0}{10/s}$$

$$\frac{1}{s+1} + 2.5 = \frac{2V_0}{10} + \frac{sV_0}{10} = \frac{1}{10}V_0(s+2)$$

$$\frac{10}{s+1} + 25 = V_0(s+2)$$

$$V_0 = \frac{25s + 35}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$



Partial-fraction decomposition



$$V_0(s) = \frac{25s + 35}{(s + 1)(s + 2)} = \frac{A}{(s + 1)} + \frac{B}{(s + 2)}$$

$$A = (s + 1) \cdot V_0(s) \Big|_{s=-1} = \frac{25s + 35}{(s + 2)} \Big|_{s=-1} = \frac{10}{1} = 10$$

$$B = (s + 2) \cdot V_0(s) \Big|_{s=-2} = \frac{25s + 35}{(s + 1)} \Big|_{s=-2} = \frac{-15}{-1} = 15$$

$$V_0(s) = \frac{10}{(s + 1)} + \frac{15}{(s + 2)}$$

$$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = (10 \cdot e^{-t} + 15 \cdot e^{-2t}) \cdot u(t) \text{ V}$$

Laplace Transform Applications

LPT.06 – Find $v_o(0)$. (Note \rightarrow since the voltage input is multiplied by $u(t)$, the voltage source is a short for all $t < 0$ and $i_L(0)=0$.)

Solution $\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a} \leftarrow \text{LPL. 01}$ $V(s) = \frac{30}{s+2}$

$$V_o(s) = V(s) \cdot \frac{2 \times 2s}{2 \times 2s + 1} = \frac{30}{s+2} \cdot \frac{4s / (2+2s)}{4s / (2+2s) + 1}$$

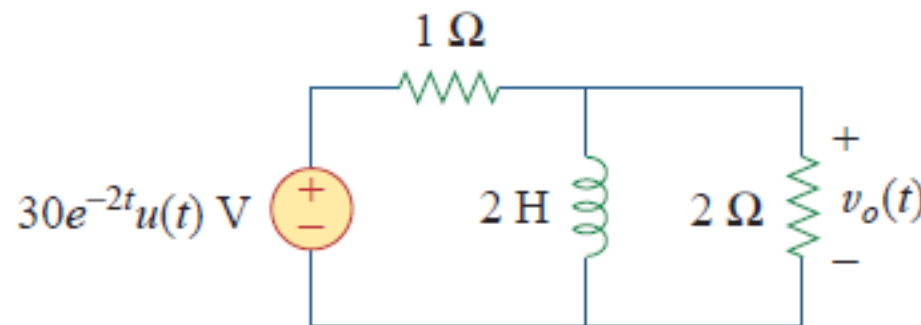
$$= \frac{30}{s+2} \cdot \frac{4s}{4s+2+2s} = \frac{30}{s+2} \cdot \frac{2s}{3s+1} = \frac{20s}{(s+2) \cdot \left(s + \frac{1}{3}\right)} = \frac{A}{s+2} + \frac{B}{\left(s + \frac{1}{3}\right)}$$

$$A = (s+2) \cdot V_o(s) \Big|_{s=-2} = \frac{20s}{\left(s + \frac{1}{3}\right)} \Big|_{s=-2} = 24$$

$$B = \left(s + \frac{1}{3}\right) \cdot V_o(s) \Big|_{s=-\frac{1}{3}} = \frac{20s}{(s+2)} \Big|_{s=-\frac{1}{3}} = -4$$

$$V_o(s) = \frac{24}{(s+2)} - \frac{4}{\left(s + \frac{1}{3}\right)}$$

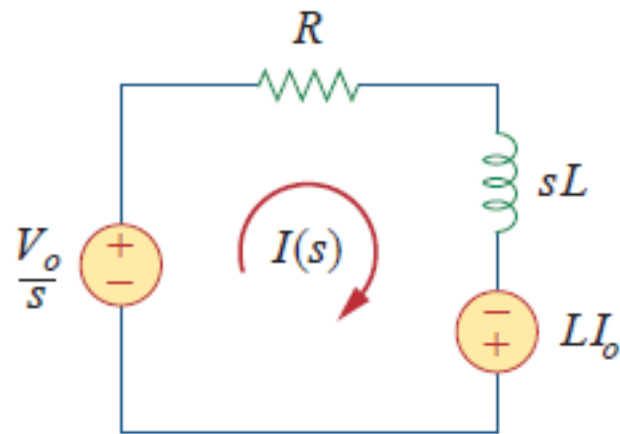
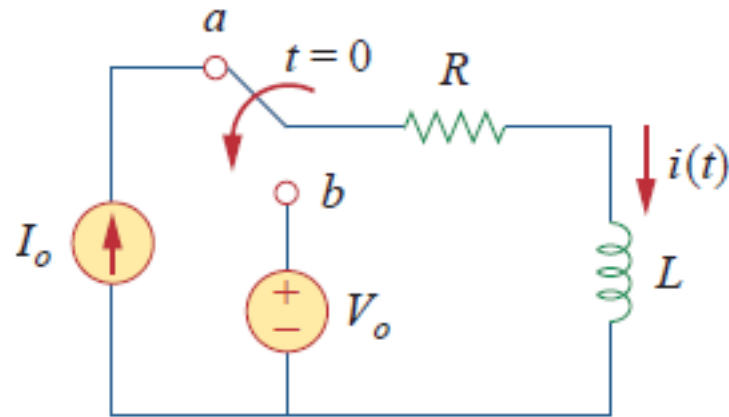
$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = \left(24 \cdot e^{-2t} - 4 \cdot e^{-\frac{t}{3}}\right) \cdot u(t) \text{ V}$$



\rightarrow Partial-fraction decomposition

Laplace Transform Applications

LPT.07 – The switch moves from position *a* to position *b* at $t = 0$. Find $i(t)$ for $t > 0$.



Solution $i(0) = I_0$

$$I(s)(R + sL) - L \cdot I_0 - \frac{V_0}{s} = 0$$

$$I(s) = L \cdot \frac{I_0}{R + sL} + \frac{V_0}{s(R + sL)} = \frac{I_0}{s + R/L} + \frac{V_0/L}{s(s + R/L)}$$

$$I(s) = \frac{I_0}{s + R/L} + \frac{V_0/R}{s} - \frac{V_0/R}{s + R/L} = \frac{I_0 - V_0/R}{s + R/L} + \frac{V_0/R}{s}$$

$$i(t) = \left(I_0 - \frac{V_0}{R} \right) e^{-t/\tau} + \frac{V_0}{R}, t \geq 0$$

$$i(t) = I_0 e^{-t/\tau} + \frac{V_0}{R} (1 - e^{-t/\tau}), t \geq 0$$

Laplace Transform Applications

LPT.08 – The switch has been in position b for a long time. It is moved to position a at $t = 0$. Determine $v(t)$ for $t > 0$.

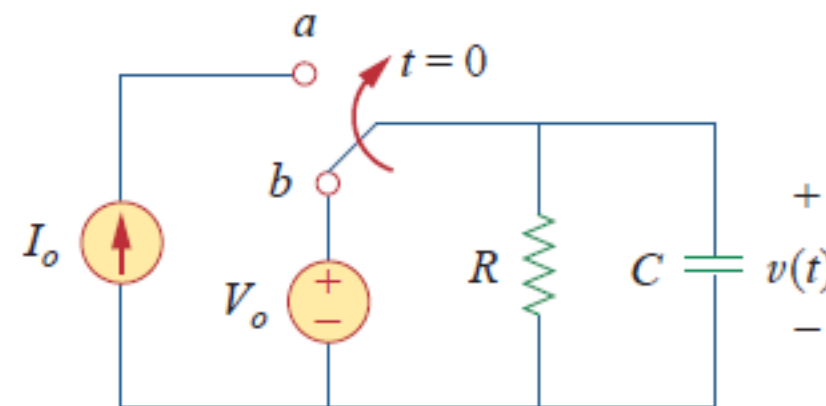
Solution *init. cond.* $\rightarrow i_C(0) = C \cdot v_0(0) = C \cdot V_0$

$$\frac{I_0}{s} + C \cdot V_0 = \frac{V(s)}{R} + s \cdot C \cdot V(s) \rightarrow \frac{I_0}{s} + C \cdot V_0 = V(s) \cdot \left[\frac{1}{R} + s \cdot C \right]$$

$$V(s) = \frac{I_0}{s \left(\frac{1}{R} + s \cdot C \right)} + \frac{C \cdot V_0}{\left(\frac{1}{R} + s \cdot C \right)} = \frac{I_0 / C}{s \cdot \left(\frac{1}{R \cdot C} + s \right)} + \frac{V_0}{\frac{1}{R \cdot C} + s}$$

$$V(s) = \frac{I_0 \cdot R}{s} - \frac{I_0 \cdot R}{\frac{1}{R \cdot C} + s} + \frac{V_0}{\frac{1}{R \cdot C} + s} = \frac{I_0 \cdot R}{s} + \frac{V_0 - I_0 \cdot R}{\frac{1}{R \cdot C} + s}$$

$$v(t) = \mathcal{L}^{-1}\{V(s)\} = I_0 \cdot R + (V_0 - I_0 \cdot R)e^{-\frac{t}{\tau}}, \quad t > 0, \quad \tau = RC$$



Laplace Transform Applications

LPT.09 – Find the voltage across the capacitor assuming that the value of $v_s(t) = 10 u(t)$ and $i_L(0) = -1 \text{ A}$, $v_C(0) = +5 \text{ V}$.

Solution

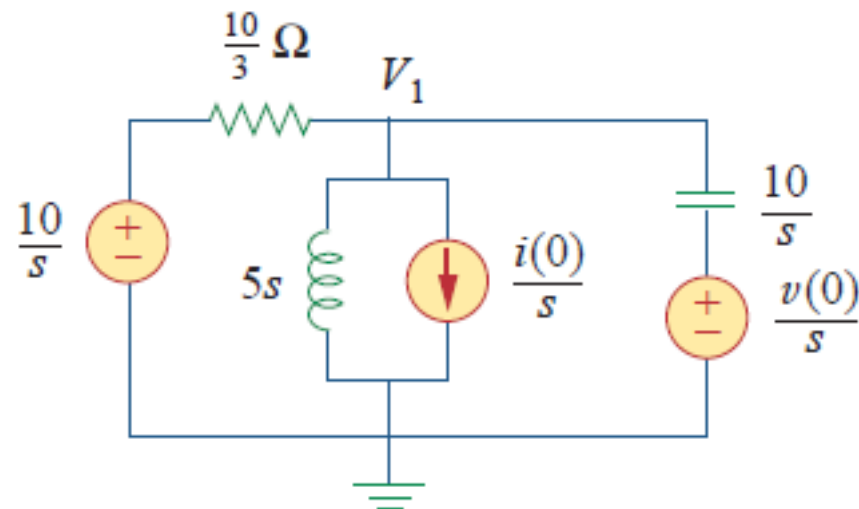
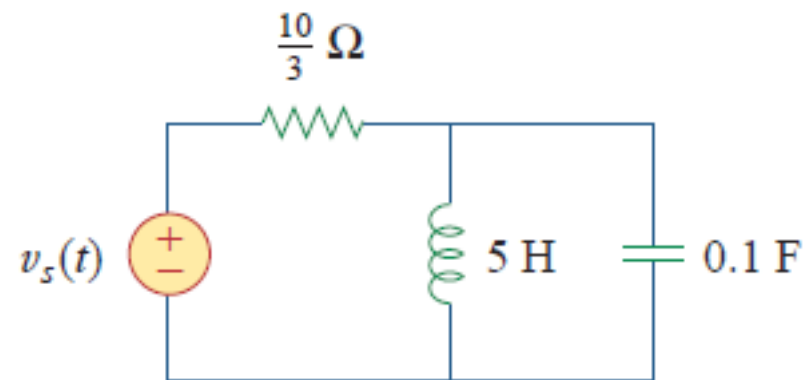
$$\frac{V_1 - 10/s}{10/3} + \frac{V_1}{5s} + \frac{i(0)}{s} + \frac{V_1 - [v(0)/s]}{10/s} = 0$$

$$0.1 \left(s + 3 + \frac{2}{s} \right) V_1 = \frac{3}{s} + \frac{1}{s} + 0.5$$

$$(s^2 + 3s + 2)V_1 = 40 + 5s$$

$$V_1 = \frac{40 + 5s}{(s + 1)(s + 2)} = \frac{35}{s + 1} - \frac{30}{s + 2}$$

$$v_1(t) = (35e^{-t} - 30e^{-2t})u(t) \text{ V}$$



Laplace Transform Applications

LPT.10 – Use superposition theorem to find the capacitor voltage. Initial conditions $i_L(0) = -1 \text{ A}$, $v_C(0) = +5 \text{ V}$

Solution

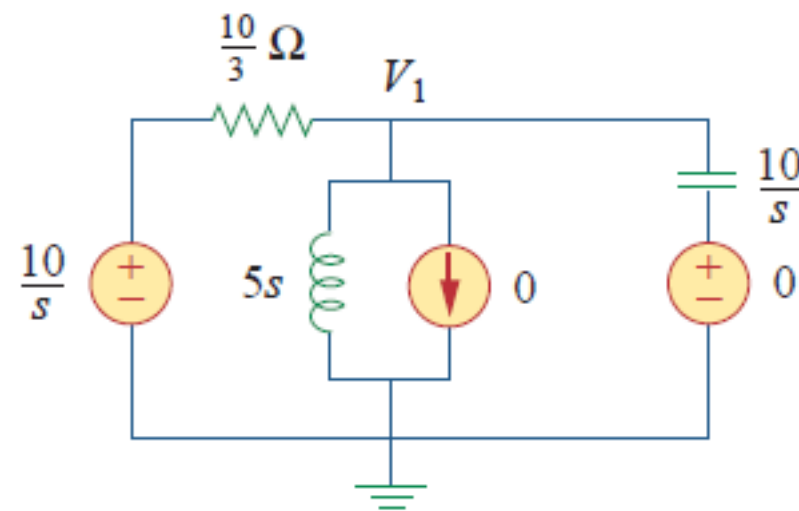
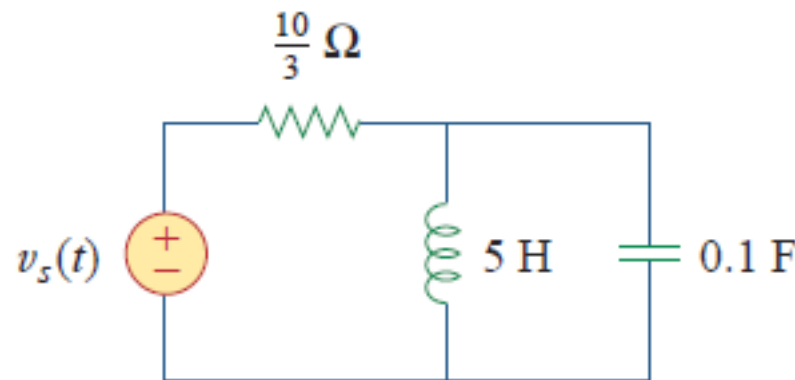
$$\frac{V_1 - 10/s}{10/3} + \frac{V_1 - 0}{5s} + 0 + \frac{V_1 - 0}{10/s} = 0$$

$$0.1 \left(s + 3 + \frac{2}{s} \right) \cdot V_1 = \frac{3}{s}$$

$$(s^2 + 3s + 2) \cdot V_1 = 30$$

$$V_1 = \frac{30}{(s+1)(s+2)} = \frac{30}{s+1} - \frac{30}{s+2}$$

$$v_1(t) = (30 \cdot e^{-t} - 30 \cdot e^{-2t}) \cdot u(t) \text{ V}$$



Laplace Transform Applications

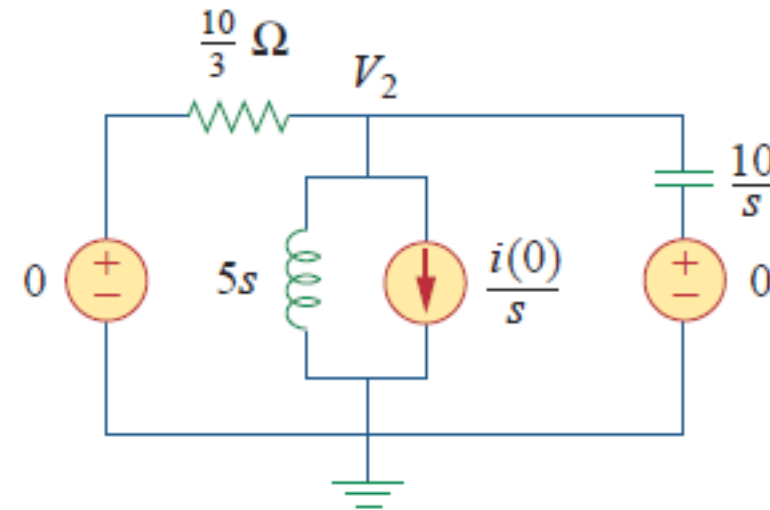
$$\frac{V_2 - 0}{10/3} + \frac{V_2 - 0}{5s} - \frac{1}{s} + \frac{V_2 - 0}{10/s} = 0$$

$$0.1 \left(s + 3 + \frac{2}{s} \right) \cdot V_2 = \frac{1}{s}$$

$$(s^2 + 3s + 2) \cdot V_2 = 30$$

$$V_2 = \frac{10}{(s+1)(s+2)} = \frac{10}{s+1} - \frac{10}{s+2}$$

$$v_2(t) = (10 \cdot e^{-t} - 10 \cdot e^{-2t}) \cdot u(t) \text{ V}$$



Laplace Transform Applications

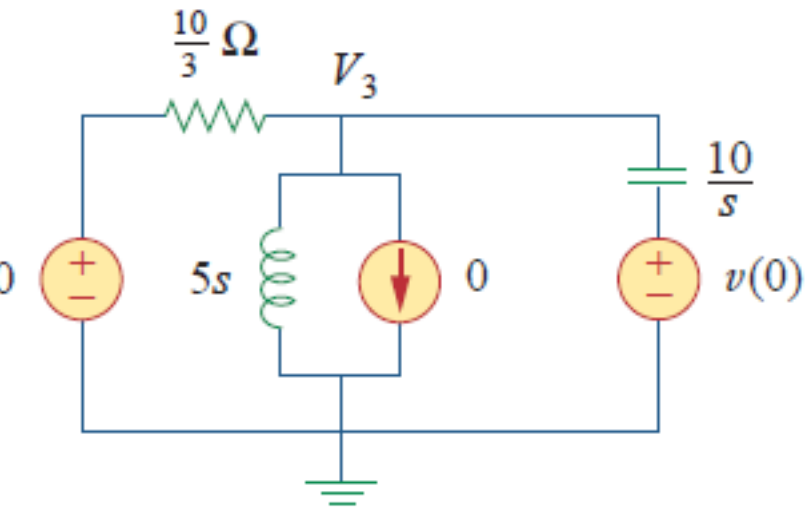
$$\frac{V_3 - 0}{10/3} + \frac{V_3 - 0}{5s} - 0 + \frac{V_3 - 5/s}{10/s} = 0$$

$$0.1 \left(s + 3 + \frac{2}{s} \right) \cdot V_3 = 0.5$$

$$V_3 = \frac{5s}{(s+1)(s+2)} = \frac{-5}{s+1} + \frac{10}{s+2}$$

$$v_3(t) = (-5 \cdot e^{-t} + 10 \cdot e^{-2t}) \cdot u(t) \text{ V}$$

$$v_1(t) = (30 \cdot e^{-t} - 30 \cdot e^{-2t}) \cdot u(t) \text{ V}$$



$$v_2(t) = (10 \cdot e^{-t} - 10 \cdot e^{-2t}) \cdot u(t) \text{ V}$$

$$v(t) = v_1(t) + v_2(t) + v_3(t) = \{(30 + 10 - 5) \cdot e^{-t} + (-30 - 10 + 10) \cdot e^{-2t}\} \cdot u(t)$$

$$v(t) = \{35 \cdot e^{-t} - 30 \cdot e^{-2t}\} \cdot u(t) \quad \leftarrow \text{(the same as it was in previous example)}$$

