



DR. GYURCSEK ISTVÁN

Exercises with Fourier Transform

Sources and additional materials (recommended)

- ❑ *Dr. Gyurcsek – Dr. Elmer: Theories in Electric Circuits, GlobeEdit, 2016, ISBN:978-3-330-71341-3*
- ❑ *Ch. Alexander, M. Sadiku: Fundamentals of Electric Circuits, 6th Ed., McGraw Hill NY 2016, ISBN: 978-0078028229*
- ❑ *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 1. MK Budapest 2002, TK szám: 49203/I*
- ❑ *Dr. Selmeczi K. – Schnöller A.: Villamosságtan 2. TK Budapest 2002, ISBN:9631026043*

Definition of Fourier Transform



FRT.01 – Find the Fourier transform of the following functions $\delta(t - t_0)$, $e^{j\omega_0 t}$, $\cos \omega_0 t$

Solution $\mathcal{F}[\delta(t - t_0)] = ?$

$$F(\omega) = \mathcal{F}[\delta(t - t_0)] = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0} \leftarrow \text{shifting property spec case: } t_0 = 0 \rightarrow \mathcal{F}[\delta(t)] = 1$$

Solution $\mathcal{F}[e^{j\omega_0 t}] = ?$ $F(\omega) = \mathcal{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0) \leftarrow \text{table! (to cut the long story short ...)}$

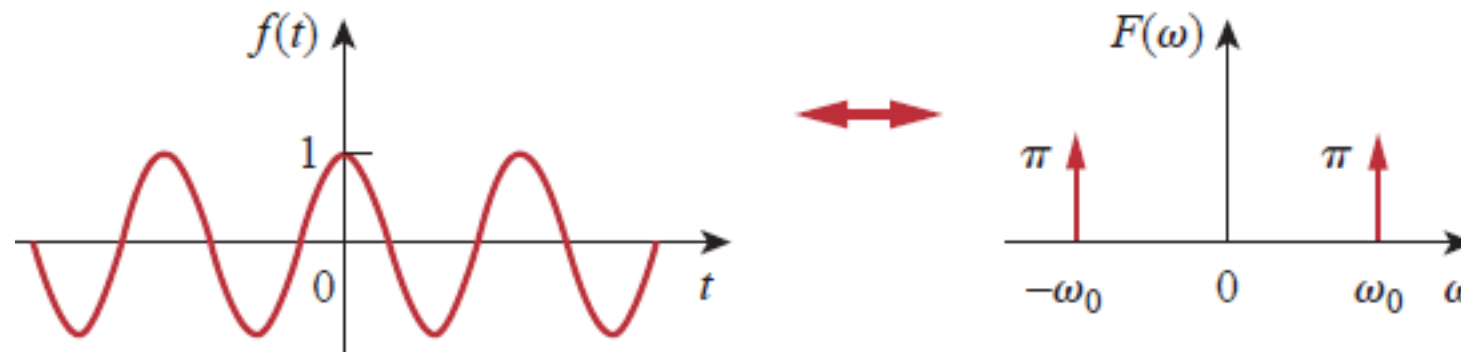
$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\text{sgn}(t)$	$\frac{2}{j\omega}$

Definition of Fourier Transform



Solution $\mathcal{F}[\cos \omega_0 t] = ?$ $\leftarrow e^x = \cos x + j \sin x$, $e^{-x} = \cos x - j \sin x$

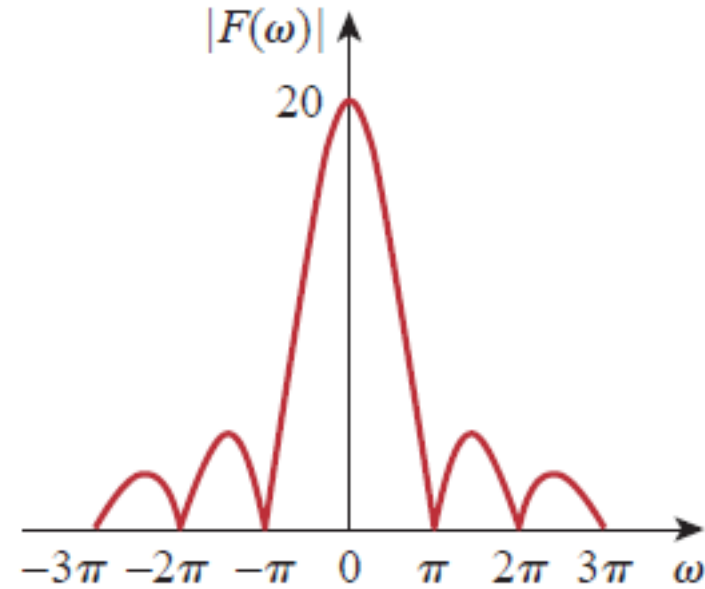
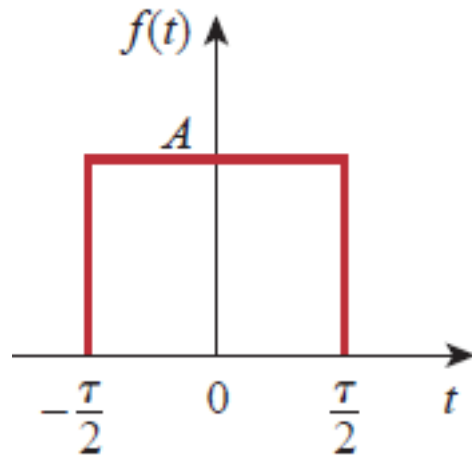
$$F(\omega) = \mathcal{F}[\cos \omega_0 t] = \mathcal{F}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] = \frac{1}{2}\mathcal{F}[e^{j\omega_0 t}] + \frac{1}{2}\mathcal{F}[e^{-j\omega_0 t}] = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



The Sinus Cardinalis Function

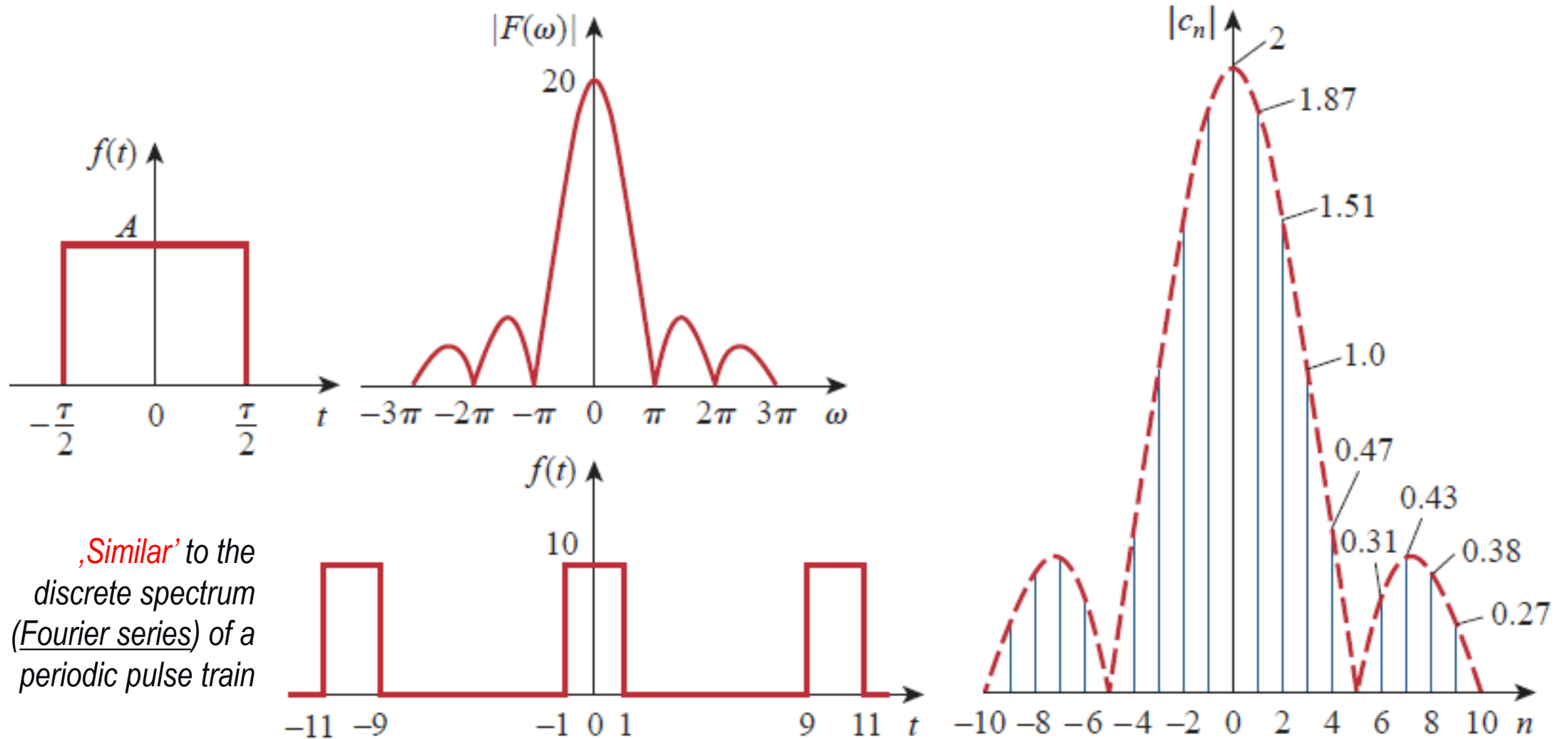


FRT.02 – Find the Fourier transform of the SINGLE rectangular pulse (*nonperiodic!*).



$$\begin{aligned} \text{Solution } F(\omega) &= \int_{-\tau/2}^{\tau/2} A e^{-j\omega t} dt = -\frac{A}{j\omega} e^{-j\omega t} \Big|_{-\tau/2}^{+\tau/2} = \frac{2A}{\omega} \left(\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j} \right) \\ &= A\tau \frac{\sin \omega\tau/2}{\omega\tau/2} = A\tau \text{ sinc } \omega\tau/2 \quad A = 10, \tau = 2 \rightarrow F(\omega) = 20 \text{ sinc } \omega \end{aligned}$$

Definition of Fourier Transform



Circuit Applications



FRT.03 – Find $v_o(t)$ for $v_i(t) = 2e^{-3t}u(t)$ V

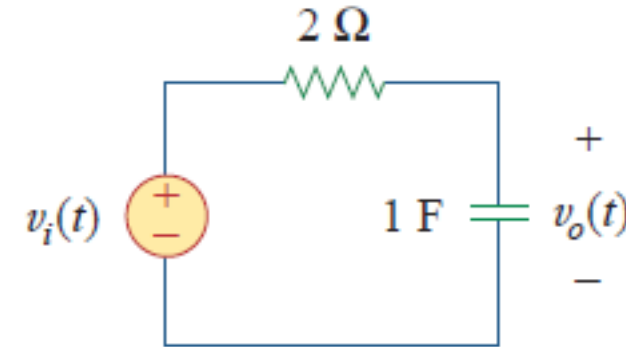
Solution
$$V_i(\omega) = \frac{2}{3 + j\omega}$$

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1/j\omega}{2 + 1/j\omega} = \frac{1}{1 + j2\omega}$$

$$V_o(\omega) = V_i(\omega)H(\omega) = \frac{2}{(3 + j\omega)(1 + j2\omega)} = \frac{1}{(3 + j\omega)(0.5 + j\omega)}$$

partial fractions
$$\rightarrow V_o(\omega) = \frac{-0.4}{3 + j\omega} + \frac{0.4}{0.5 + j\omega}$$

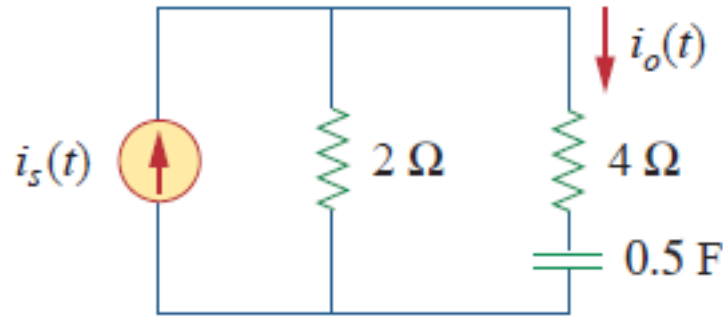
$$\mathcal{F}^{-1}[V_o(\omega)] = 0.4(e^{-0.5t} - e^{-3t})u(t)$$



Circuit Applications



FRT.04 – Find $i_o(t)$ for $i_s(t) = 10 \sin 2t$ A



Solution $I_S(\omega) = j\pi 10[\delta(\omega + 2) - \delta(\omega - 2)]$

$$H(\omega) = \frac{I_o(\omega)}{I_S(\omega)} = \frac{2}{2 + 4 + 2/j\omega} = \frac{j\omega}{1 + j\omega 3}$$

$$I_o(\omega) = H(\omega)I_S(\omega) = \frac{10\pi\omega[\delta(\omega - 2) - \delta(\omega + 2)]}{1 + j\omega 3}$$

$$i_o(t) = \mathcal{F}^{-1}[I_o(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{10\pi\omega[\delta(\omega - 2) - \delta(\omega + 2)]}{1 + j\omega 3} e^{j\omega t} d\omega$$

shifting property $\rightarrow \int_{-\infty}^{\infty} \delta(\omega - \omega_0) f(\omega) d\omega = f(\omega_0)$

$$i_o(t) = \frac{10\pi}{2\pi} \left[\frac{2}{1 + j6} e^{j2t} - \frac{-2}{1 - j6} e^{-j2t} \right] = 10 \left[\frac{e^{j2t}}{6.082 e^{j80.54^\circ}} - \frac{e^{-j2t}}{6.082 e^{-j80.54^\circ}} \right] = 1.644 [e^{j(2t-80.54^\circ)} + e^{-j(2t-80.54^\circ)}]$$

$$i_o(t) = 3.288 \cos(2t - 80.54^\circ) \text{ A}$$

Parseval's Theorem



FRT.05 – The voltage across a 10-Ω resistor is $v(t) = 5e^{-3t}u(t)$ V. Find the total energy dissipated in the resistor

Solution 1 (in time domain)
$$W_{10\Omega} = \frac{1}{10} \int_{-\infty}^{\infty} v^2(t) dt = 0.1 \int_0^{\infty} 25 e^{-6t} dt = 2.5 \left. \frac{e^{-6t}}{-6} \right|_0^{\infty} = \frac{2.5}{6} = 416.7 \text{ mJ}$$

Solution 2 (in frequency domain)
$$V(\omega) = \frac{5}{3 + j\omega} \rightarrow |V(\omega)|^2 = V(\omega)V(\omega)^* = \frac{25}{9 + \omega^2}$$

$$W_{10\Omega} = \frac{0.1}{2\pi} \int_{-\infty}^{\infty} |V(\omega)|^2 d\omega = \frac{0.1}{\pi} \int_0^{\infty} \frac{25}{9 + \omega^2} d\omega = \frac{2.5}{\pi} \left(\frac{1}{3} \tan^{-1} \frac{\omega}{3} \right) \Big|_0^{\infty} = \frac{2.5}{\pi} \cdot \frac{1}{3} \cdot \frac{\pi}{2} = \frac{2.5}{6} = 416.7 \text{ mJ}$$

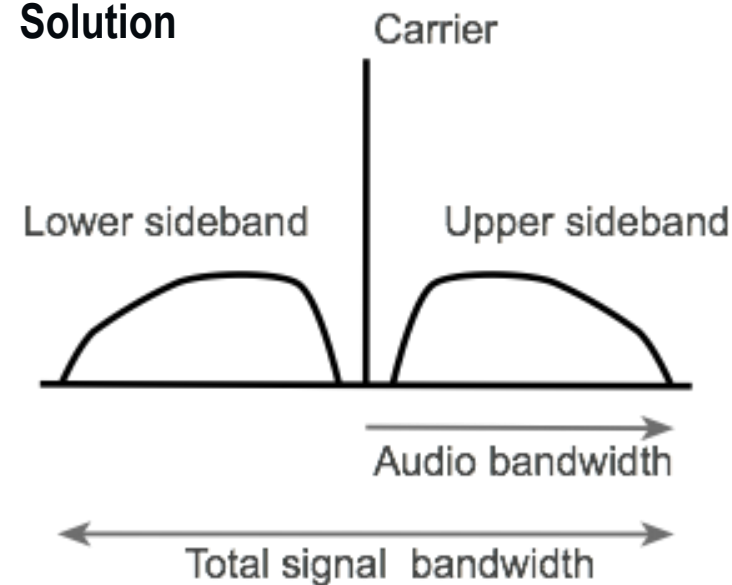
$$\text{applying} \rightarrow \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Amplitude Modulation



FRT.06 – A music signal has frequency components from 15 Hz to 30 kHz and this signal is used to amplitude modulate a 1.2-MHz carrier. Find the range of frequencies for the lower and upper sidebands.

Solution



$$f_{LSB1} = 1,200,000 - 30,000 = 1,170,000 \text{ Hz}$$

$$f_{LSB2} = 1,200,000 - 15 = 1,199,985 \text{ Hz}$$

$$f_{USB3} = 1,200,000 + 15 = 1,200,015 \text{ Hz}$$

$$f_{USB4} = 1,200,000 + 30,000 = 1,230,000 \text{ Hz}$$

Nyquist–Shannon Sampling Theorem



FRT.07 – A telephone signal with a cutoff frequency of 5 kHz is sampled at a rate 60 percent higher than the minimum allowed rate. Find the sampling rate.

Solution *Nyquist rate* $\rightarrow f_{S \min} = 2 \cdot f_{\text{cut-off}} = 2 \cdot 5 = 10 \text{ kHz}$

$$f_S = 1.6 \cdot f_{S \min} = 1.6 \cdot 10 = 16 \text{ kHz}$$

Questions

